DE LA RECHERCHE À L'INDUSTRIE

A conservative slide-line method for cell-centered semi-Lagrangian and ALE schemes in 2D

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Introduction

Objectives

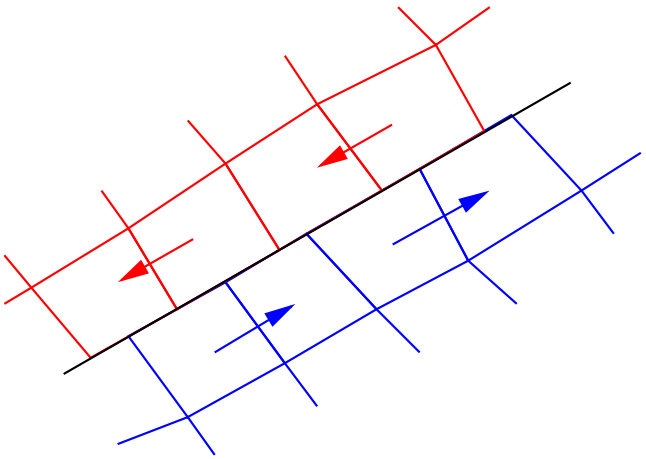
- Sliding of two compressible fluid domains
- **Lagrangian** conservative and stable method
  - Use of finite-volume like schemes (nodal solvers)
    - Glace Després, Mazeran *Arch. Rationnal Mech. Anal.*, 2005
    - Eucclhyd Maire, Abgrall, Breil, Ovadia *SIAM J. Sci. Comput.*, 2007
  - Lack of **Lagrangian** conservative methods in literature
    Conservation defects are often a measure of the approximation quality
    eg: Kucharik, Loubère, Bednárik, Liska *Comput. Fluid.*, 2012
  - Parallel study ("discrete" approach + contacts management)
    Clair, Després, Labourasse (see G. Clair's presentation)
- Compatibility with (S)ALE formulations (without mixing)

Difficulties (*a priori*)

- Discrete interface (geometry) not defined properly
- Conservative and stable method

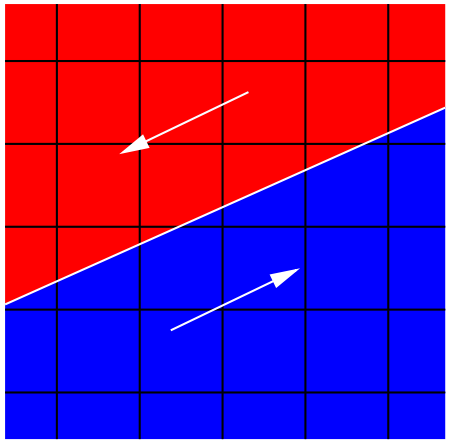
Illustrations and classical approaches

Lagrangian approach



- Each fluid domain is defined by its own mesh

Eulerian approach



- Interface reconstruction between fluids
- ρ, $\mathbf{u}$ and ϵ are defined in each material of sliding cells (mixed cells)

Continuity of normal velocity is imposed at the interface (in a consistent way with regard to Euler's equations)

Few references

- Very rich literature in the contact/sliding field
  Bourago, Kukudzhanov *Izv. RAN MTT No. 1*, 2005 (more than 600 ref.)
- Compressible solvers: gas/fluid/elasto-plastic materials
  - Eulerian
    Braeunig, Desjardins, Ghidaglia *Eur. J. Mech. B-Fluids*, 2007
    Claisse, Ghidaglia, Rouzier under preparation (see A. Claisse's poster))
  - ALE
    Folzan *Ph. D. Thesis* (Le Tallec, Perlat), 2013
  - Lagrangian
    Wilkins *Meth. Comput. Phys.*, 1964
    Caramana *J. Comput. Phys.*, 2009
    Kucharik, Loubère, Bednárik, Liska *Comput. Fluid.*, 2012
    Morgan, Kenamond, Burton, Carney, Ingraham *J. Comput. Phys.*, 2013
    Clair, Després, Labourasse ongoing publication

Euler equations

Integral forms (formal but suited to finite-volume schemes derivation)

Lagrangian formulation

$\forall \omega(t)$ Lagrangian in Ω

$$\frac{d}{dt}\int_{\omega(t)} 1 = \int_{\partial\omega(t)} \mathbf{u}\cdot\mathbf{n}$$

$$\frac{d}{dt}\int_{\omega(t)} \rho = 0$$

$$\frac{d}{dt}\int_{\omega(t)} \rho\mathbf{u} = -\int_{\partial\omega(t)} p\,\mathbf{n}$$

$$\frac{d}{dt}\int_{\omega(t)} \rho E = -\int_{\partial\omega(t)} p\mathbf{u}\cdot\mathbf{n}$$

Eulerian formulation

$\forall \omega$ fixed in Ω

$$\frac{d}{dt}\int_{\omega} 1 = 0$$

$$\frac{d}{dt}\int_{\omega} \rho = -\int_{\partial\omega} \rho\mathbf{u}\cdot\mathbf{n}$$

$$\frac{d}{dt}\int_{\omega} \rho\mathbf{u} = -\int_{\partial\omega} (\rho\mathbf{u}\otimes\mathbf{u} + p\mathrm{I})\,\mathbf{n}$$

$$\frac{d}{dt}\int_{\omega} \rho E = -\int_{\partial\omega} (\rho E + p)\mathbf{u}\cdot\mathbf{n}$$

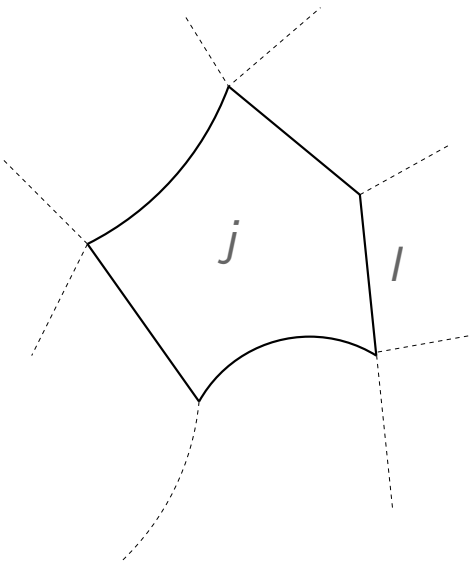
with $p = p(\rho,\epsilon), \quad \epsilon = E - \frac{1}{2}\mathbf{u}\cdot\mathbf{u}$

Outline

1. Abstract scheme without slide-lines
2. $\mathbb{P}_1$ discretization without slide-lines
3. Abstract scheme with slide-lines
4. $\mathbb{P}_1 - \mathbb{P}_0$ discretization with slide-lines
5. Numerical tests
6. Conclusions and perspectives

Notations and scheme structure

- Scheme is based on SDP *C. R. Acad. Sci. Paris, Ser. I*, 2010
- Notations
  - Let $\mathcal{M}$, a conformal grid of $\Omega(t)$
  - Any edge $l = \mathbf{T}_l([0,1])$ where $\mathbf{T}_l$ is bijective on $[0,1]$ and smooth enough
  - $\mathbf{u}_j(\mathbf{x})$ and $p_j(\mathbf{x})$ denote the reconstructions at time t supported by cell j
  - $\mathcal{L}_j$ is the set of edges l of cell j
- Scheme structure



$$\forall j, \qquad \frac{d}{dt}\int_j 1 = \sum_{l\in\mathcal{L}_j}\int_l \mathbf{u}^\star\cdot\mathbf{n}_{jl},$$

$$\frac{d}{dt}\int_j \rho = 0,$$

$$\frac{d}{dt}\int_j \rho\mathbf{u} = -\sum_{l\in\mathcal{L}_j}\int_l p_j^\star\,\mathbf{n}_{jl},$$

$$\frac{d}{dt}\int_j \rho E = -\sum_{l\in\mathcal{L}_j}\int_l p_j^\star\,\mathbf{u}^\star\cdot\mathbf{n}_{jl}.$$

Objective: compute $\mathbf{u}^\star$ and $p_j^\star$

Calculation of $\mathbf{u}^\star$ and $p_j^\star$'s

Acoustic solver ($dp + \rho c d\mathbf{u} \cdot n = 0$)

The relation is written at each point of each edge l, along the local outgoing normal to cell j

$$\forall j, \forall l \in \mathcal{L}_j, \forall \mathbf{x} \in l, \qquad p_j^\star(\mathbf{x}) - p_j(\mathbf{x}) + (\rho c)_j \left(\mathbf{u}^\star(\mathbf{x}) - \mathbf{u}_j(\mathbf{x})\right) \cdot \mathbf{n}_{jl}(\mathbf{x}) = 0.$$

Introducing $A_{jl}(\mathbf{x}) := (\rho c)_j \mathbf{n}_{jl} \otimes \mathbf{n}_{jl}(\mathbf{x})$ and $\mathcal{E} := \{\mathbf{x} \in \mathbb{R}^d / \exists l \text{ t.q. } \mathbf{x} \in l\}$, one gets the following **weak form**

$$\forall j,\ \forall l \in \mathcal{L}_j,\ \forall \mathbf{v} \in L^2(\mathcal{E})^d, \quad \int_l p_j^\star \mathbf{n}_{jl} \cdot \mathbf{v} + \int_l {}^t\mathbf{v} A_{jl}\, \mathbf{u}^\star = \int_l p_j \mathbf{n}_{jl} \cdot \mathbf{v} + \int_l {}^t\mathbf{v} A_{jl}\, \mathbf{u}_j$$

Conservation constrain

$$\forall \mathbf{v} \in L^2(\mathcal{E})^d, \quad \sum_j \sum_{l \in \mathcal{L}_j} \int_l p_j^\star \mathbf{n}_{jl} \cdot \mathbf{v} = 0$$

Remark: Weak equality of pressures on both sides of the edges are imposed. This distinction seems necessary to derive the scheme.

Calculation of $\mathbf{u}^\star$

$$\forall \mathbf{v} \in L^2(\mathcal{E})^d, \quad \sum_j \sum_{l \in \mathcal{L}_j} \int_l {}^t\mathbf{v}\, A_{jl}\, \mathbf{u}^\star = \sum_j \sum_{l \in \mathcal{L}_j} \int_l p_j \mathbf{n}_{jl} \cdot \mathbf{v} + \sum_j \sum_{l \in \mathcal{L}_j} \int_l {}^t\mathbf{v}\, A_{jl}\, \mathbf{u}_j$$

$$a(\mathbf{u}^\star, \mathbf{v}) = l(\mathbf{v})$$

$$J_{\mathbf{u}^\star} = \inf_{\mathbf{v}} J_{\mathbf{v}} \quad \text{where} \quad J_{\mathbf{v}} = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - l(\mathbf{v})$$

Existence and uniqueness of $\mathbf{u}^\star$?

- a is not coercive on $L^2(\mathcal{E})^d$: non uniqueness of tangential velocities (expected)
- a is coercive on $\mathcal{N} := \{\mathbf{w} \in L^2(\mathcal{E})^d,\ \text{t.q. } \forall l, \mathbf{w}|_l = w_l \mathbf{n}_l,\ \text{où } w_l \in L^2(l)\}$
  - $\mathcal{N}$, **the space of normal velocities to the edges**, is the natural space for this problem.
  - How to discretize it in a conformal way?
  - Not important here for the following (sliding), one proceeds with $L^2(\mathcal{E})^d$

Scheme properties

The scheme is conservative

- Volume: $\sum_j \sum_{l \in \mathcal{L}_j} \int_l \mathbf{u}^\star \cdot \mathbf{n}_{jl} = \sum_l \int_l \left(\mathbf{u}^\star \cdot \sum_{j \in \mathcal{J}_l} \mathbf{n}_{jl} \right) = 0$
- Mass: null fluxes by construction
- Momentum: $\sum_j \sum_{l \in \mathcal{L}_j} \int_l p_j^\star \mathbf{n}_{jl} = \sum_l \sum_{j \in \mathcal{J}_l} \int_l p_j^\star \mathbf{n}_{jl} = \mathbf{0}$
- Total energy: $\sum_j \sum_{l \in \mathcal{L}_j} \int_l p_j^\star \mathbf{u}^\star \cdot \mathbf{n}_{jl} = \sum_l \sum_{j \in \mathcal{J}_l} \int_l p_j^\star \mathbf{u}^\star \cdot \mathbf{n}_{jl} = 0$

The continuous in time scheme is entropic (for constant data in cells)

Let us recall that $\int_{\omega(t)} \rho \frac{d}{dt} \psi = \frac{d}{dt} \int_{\omega(t)} \rho \psi$

$$
\begin{aligned}
V_j \rho_j T_j \frac{d}{dt} S_j &= V_j \rho_j \frac{d}{dt} e_j + V_j \rho_j p_j \frac{d}{dt} \frac{1}{\rho_j} \\
&= \frac{d}{dt} \int_j \rho_j E_j - \mathbf{u}_j \cdot \frac{d}{dt} \int_j \rho_j \mathbf{u}_j + p_j \frac{d}{dt} \int_j 1 \\
&= \sum_{l \in \mathcal{L}_j} \left(\int_l^t (\mathbf{u}_j - \mathbf{u}^\star) A_{jl} (\mathbf{u}_j - \mathbf{u}^\star) \right) \geq 0
\end{aligned}
$$

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5 Numerical tests

6 Conclusions and perspectives

$\mathbb{P}_1$ discretization of $\mathbf{u}^\star$

Discrete problem

- hypothesis: $\mathcal{E}$ set of straight edges
- $\mathbb{V}_h = \mathbb{P}_1(\mathcal{E})^d \subset L^2(\mathcal{E})^d \qquad (\mathbb{P}_1(\mathcal{E})^d \implies$ continuous mesh displacement$)$
- Find $\mathbf{u}_h^\star \in \mathbb{V}_h$ such that $\quad \forall \mathbf{v}_h \in \mathbb{V}_h, \quad a(\mathbf{u}_h^\star, \mathbf{v}_h) = l(\mathbf{v}_h)$

Properties

- $\mathbb{V}_h \subset L^2(\mathcal{E})^d \implies$ conservative and entropic
- a is coercive on $\mathbb{V}_h \implies$ well-posed problem
- All properties remain true with numerical quadrature

Resolution

- Global linear system
  - all velocity degrees of freedom are coupled
  - L^2-projection matrix (good condition number, obvious preconditioners)
- Trapezium formula: mass lumping $\implies$ Eucclhyd

Remark: a $\mathbb{P}_0$ discretization of $\mathbf{u}^\star$ the edges dual mesh $\implies$ Glace
i.e. $\mathbf{u}_h^\star$ is constant on the half-edges connected to a vertex

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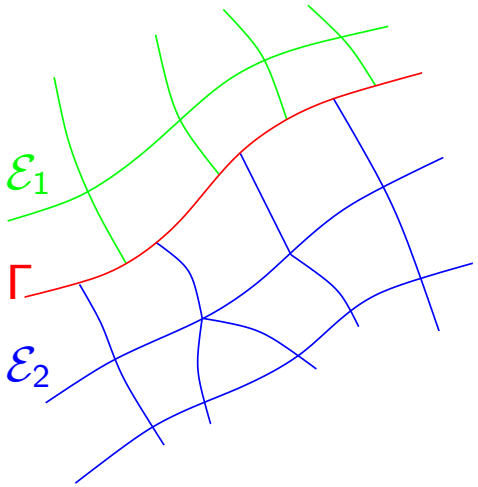
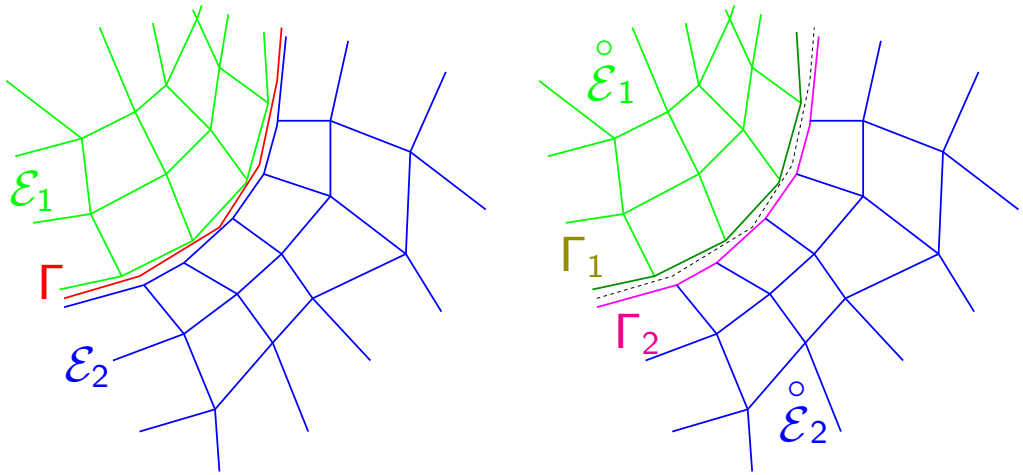
Abstract scheme with slide-lines

"Ideal" configuration



Practical configuration



- Scheme without sliding: $\mathbf{u}^\star$ is solution of a minimization problem
- Sliding: $[\![\mathbf{u}^\star]\!] \cdot \mathbf{n} = 0$ at the interface on non-conformal grids
- Mortar-like approach (Bernardi, Maday, Patera)

$$\forall \mu \in L^2(\Gamma), \quad \int_\Gamma (\mathbf{u}_1^\star - \mathbf{u}_2^\star) \cdot \mathbf{n}\, \mu = 0.$$

- Find $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) \in \mathcal{C}$, where

$$\mathcal{C} := \left\{ (\mathbf{v}_1, \mathbf{v}_2) \in L^2(\mathcal{E}_1)^d \times L^2(\mathcal{E}_2)^d / \ \forall \mu \in L^2(\Gamma), \ \int_\Gamma (\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n}\, \mu = 0 \right\}$$

Scheme structure

Let $\mathbf{n}_{il}$ the unit normal to Γ, outgoing from Ω_i "seen" by the edge l

Scheme structure

$$\forall i,\ \forall j \in \mathcal{M}_i, \qquad \frac{d}{dt}\int_j 1 = \sum_{l\in\mathcal{L}_j}\int_l \mathbf{u}_i^\star \cdot \mathbf{n}_{jl}$$

$$\frac{d}{dt}\int_j \rho = 0$$

$$\frac{d}{dt}\int_j \rho\mathbf{u} = -\sum_{\substack{l\in\mathcal{L}_j\\ l\notin\Gamma_i}}\int_l p_j^\star \mathbf{n}_{jl} - \sum_{\substack{l\in\mathcal{L}_j\\ l\subset\Gamma_i}}\int_l p_j^\star \mathbf{n}_{il}$$

$$\frac{d}{dt}\int_j \rho E = -\sum_{\substack{l\in\mathcal{L}_j\\ l\notin\Gamma_i}}\int_l p_j^\star \mathbf{u}_i^\star \cdot \mathbf{n}_{jl} - \sum_{\substack{l\in\mathcal{L}_j\\ l\subset\Gamma_i}}\int_l p_j^\star \mathbf{u}_i^\star \cdot \mathbf{n}_{il}$$

Why not using the conservative form $\frac{d}{dt}\int_j 1 = \sum_{\substack{l\in\mathcal{L}_j\\ l\notin\Gamma_i}}\int_l \mathbf{u}_i^\star \cdot \mathbf{n}_{jl} + \sum_{\substack{l\in\mathcal{L}_j\\ l\subset\Gamma_i}}\int_l \mathbf{u}_i^\star \cdot \mathbf{n}_{il}$?

- $\mathbf{u}_i^\star$ is $\mathcal{E}_i$'s velocity, not Γ's one (corresponds to the true volume variation)
- Time discretization $\rightarrow$ volumes are recomputed (not updated with fluxes)

Calculation of $\mathbf{u}_i^\star$'s and $p_j^\star$'s

Acoustic solver

- standard edges: $\forall i,\ \forall j \in \mathcal{M}_i, \forall l \in \mathcal{L}_j /\ l \not\subset \Gamma_i, \forall \mathbf{v}_j \in L^2(l)^d$

$$\int_l p_j^\star \mathbf{v}_i \cdot \mathbf{n}_{jl} + \int_l {}^t\mathbf{v}_i\, A_{jl}\, \mathbf{u}_i^\star = \int_l p_j \mathbf{v}_i \cdot \mathbf{n}_{jl} + \int_l {}^t\mathbf{v}_i\, A_{jl}\, \mathbf{u}_j$$

- interface edges: $\forall i,\ \forall j \in \mathcal{M}_i, \forall l \in \mathcal{L}_j /\ l \subset \Gamma_i, \forall \mathbf{v}_j \in L^2(l)^d$

$$\int_l p_j^\star \mathbf{v}_i \cdot \mathbf{n}_{il} + \int_l {}^t\mathbf{v}_i\, A_{il}\, \mathbf{u}_i^\star = \int_l p_j \mathbf{v}_i \cdot \mathbf{n}_{il} + \int_l {}^t\mathbf{v}_i\, A_{il}\, \mathbf{u}_j$$

with $A_{il} = (\rho c)_j \mathbf{n}_{il} \otimes \mathbf{n}_{il}$

Conservation constrain

$$\forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{C}, \quad \sum_i \left(\sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \not\subset \Gamma_i}} \int_l p_j^\star \mathbf{v}_i \cdot \mathbf{n}_{jl} + \sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \int_l p_j^\star \mathbf{v}_i \cdot \mathbf{n}_{il} \right) = 0$$

Calculation of $(\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ (1/2)

Problem in the constrained space

Find $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) \in \mathcal{C}$ s.t. $\forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{C}$

$$\sum_i \left(\sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \not\subset \Gamma_i}} \int_l {}^t\mathbf{v}_i \, A_{jl} \, \mathbf{u}_i^\star + \sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \int_l {}^t\mathbf{v}_i \, A_{il} \, \mathbf{u}_i^\star \right)$$

$$= \sum_i \left(\sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \not\subset \Gamma_i}} \left(\int_l p_j \mathbf{v}_i \cdot \mathbf{n}_{jl} + \int_l {}^t\mathbf{v}_i \, A_{jl} \, \mathbf{u}_j \right) \right.$$

$$\left. + \sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \left(\int_l p_j \mathbf{v}_i \cdot \mathbf{n}_{il} + \int_l {}^t\mathbf{v}_i \, A_{il} \, \mathbf{u}_j \right) \right)$$

That rewrites, find $(\mathbf{u}_1^\star, \mathbf{u}_2^\star) \in \mathcal{C}$ s.t. $\forall (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{C}$

$$\sum_i a_i(\mathbf{u}_i^\star, \mathbf{v}_i) = \sum_i l_i(\mathbf{v}_i)$$

Calculation of $(\mathbf{u}_1^\star, \mathbf{u}_2^\star)$ (2/2)

Saddle point problem

Find $(\mathbf{u}_1^\star, \mathbf{u}_2^\star, \lambda) \in \mathcal{V}$ s.t. $\forall (\mathbf{v}_1, \mathbf{v}_2, \mu) \in \mathcal{V}$

$$\left| \begin{array}{l} \displaystyle\sum_i a_i(\mathbf{u}_i^\star, \mathbf{v}_i) + \sum_i b_i(\mathbf{v}_i, \lambda) = \sum_i l_i(\mathbf{v}_i) \\ \qquad\qquad \text{and} \quad \displaystyle\sum_i b_i(\mathbf{u}_i^\star, \mu) = 0 \end{array} \right.$$

where $\mathcal{V} := L^2(\mathcal{E}_1)^d \times L^2(\mathcal{E}_2)^d \times L^2(\Gamma)$ and

$$\forall (\mathbf{v}_i, \mu) \in L^2(\mathcal{E}_i) \times L^2(\Gamma), \quad b_i(\mathbf{v}_i, \mu) := \sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \int_l \mathbf{v_i} \cdot \mathbf{n}_{il}\, \mu$$

Interpretation of λ

λ acts as the mean pressure imposed by one domain to the other to ensure sliding

Scheme properties

- The scheme is conservative in mass, momentum and total energy
- It is not conservative in volume at the interface

$$\sum_i \sum_{j \in \mathcal{M}_i} \frac{d}{dt} \int_j 1 = \sum_i \sum_{j \in \mathcal{M}_i} \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \int_l \mathbf{u}_i^\star \cdot (\mathbf{n}_{jl} - \mathbf{n}_{il})$$

- The scheme is invariant to Galilean frame change
- The scheme is entropic if one has in each cell j

$$p_j \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \int_l (\mathbf{u}_i^\star - \mathbf{u}_j) \cdot (\mathbf{n}_{il} - \mathbf{n}_{jl}) \leq \sum_{\substack{l \in \mathcal{L}_j \\ l \not\subset \Gamma_i}} \int_l {}^t(\mathbf{u}_j - \mathbf{u}_i^\star) A_{jl} (\mathbf{u}_j - \mathbf{u}_i^\star)$$

$$+ \sum_{\substack{l \in \mathcal{L}_j \\ l \subset \Gamma_i}} \int_l {}^t(\mathbf{u}_j - \mathbf{u}_i^\star) A_{il} (\mathbf{u}_j - \mathbf{u}_i^\star)$$

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$\mathbb{P}_1 - \mathbb{P}_0$ discretization of the velocity problem

One chooses $\mathcal{V}_h = \mathbb{P}_1(\mathcal{E}_1)^d \times \mathbb{P}_1(\mathcal{E}_2)^d \times \mathbb{P}_0(\Gamma) \subset \mathcal{V}$

Discrete problem

Find $(\mathbf{u}^\star_{1\,h}, \mathbf{u}^\star_{2\,h}, \lambda_h) \in \mathcal{V}_h$ s.t. $\forall (\mathbf{v}_{1\,h}, \mathbf{v}_{2\,h}, \mu_h) \in \mathcal{V}_h$

$$\left| \begin{array}{r} \displaystyle\sum_i a_i(\mathbf{u}^\star_{i\,h}, \mathbf{v}_{i\,h}) + \sum_i b_i(\mathbf{v}_{i\,h}, \lambda_h) = \sum_i l_i(\mathbf{v}_{i\,h}) \\ \text{and} \quad \displaystyle\sum_i b_i(\mathbf{u}^\star_{i\,h}, \mu_h) = 0 \end{array} \right.$$

Properties

- Since $\mathbb{V}_h \subset \mathbb{V}$ above properties remain true

Question

- Is the saddle-point problem well-posed?

Numerical analysis of the discrete problem

Theorem (inf-sup)

If Γ's mesh is locally coarser than the finest of Γ_i's meshes, then one has

$$\inf_{\lambda_h \in \mathbb{P}_0} \sup_{(\mathbf{u}_1,\mathbf{u}_2)\in\mathbb{P}_1(\mathcal{E}_1)^d\times\mathbb{P}_1(\mathcal{E}_2)^d} \frac{\sum_i b_i(\mathbf{v}_{i\,h},\lambda_h)}{\sum_i \|\mathbf{v}_{i\,h}\|_{0,\mathcal{E}_i}\|\lambda_h\|_{0,\Gamma}} \gtrsim 1$$

The discrete problem is well-posed under this hypothesis

Theorem (*a priori* estimates)

Under same conditions, one has

$$\sum_i \|\mathbf{u}_{i\,h}^\star\|_{0,\mathcal{E}_i} + \|\lambda_h\|_{0,\Gamma} \lesssim \sum_i \|l_i\|_{L^2(\mathcal{E}_i)'}$$

Linear System

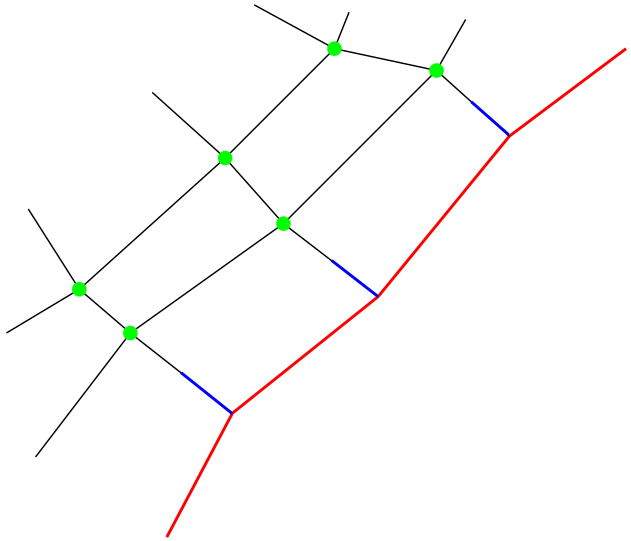
- One has to solve

$$\begin{pmatrix} A_1 & 0 & {}^tB_1 \\ 0 & A_2 & {}^tB_2 \\ B_1 & B_2 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \Lambda \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ 0 \end{pmatrix}$$

- Trapezium quadrature formula for standard edges ($\Longrightarrow$ mass lumping)

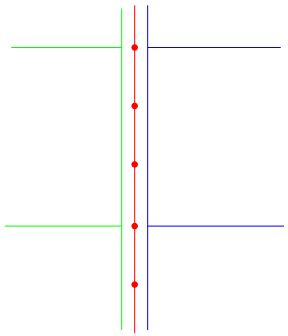
$$D_{\overset{\circ}{\mathcal{E}}_i} U_{\overset{\circ}{\mathcal{E}}_i} = L_{\overset{\circ}{\mathcal{E}}_i}, \qquad i \in \{1, 2\}$$

$$\begin{pmatrix} A_{\Gamma_1} & 0 & {}^tB_{\Gamma_1} \\ 0 & A_{\Gamma_2} & {}^tB_{\Gamma_2} \\ B_{\Gamma_1} & B_{\Gamma_2} & 0 \end{pmatrix} \begin{pmatrix} U_{\Gamma_1} \\ U_{\Gamma_2} \\ \Lambda \end{pmatrix} = \begin{pmatrix} L_{\Gamma_1} \\ L_{\Gamma_2} \\ 0 \end{pmatrix}$$



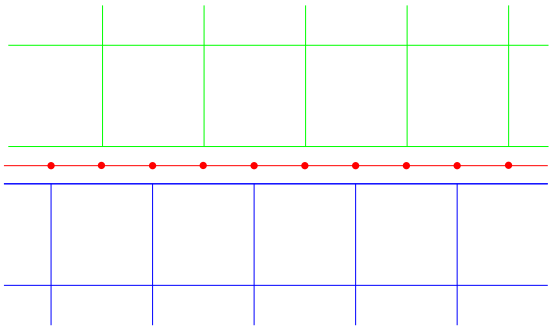
- classic nodal solver
- exact integration
- Eucclhyd contributions to the system

Pathologies



System is not invertible

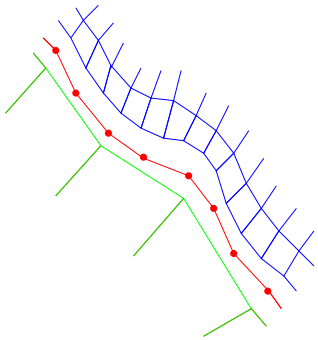
inf-sup **is not** satisfied



Numeric locking

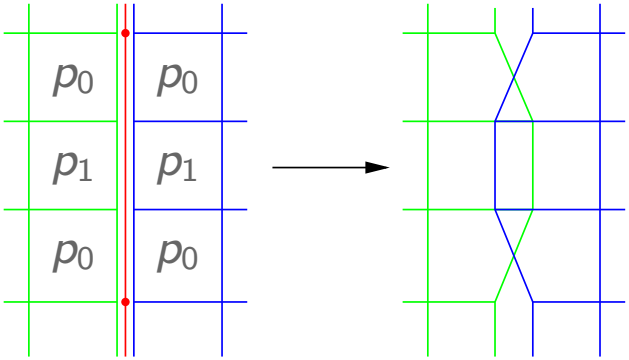
non-local velocity coupling

inf-sup **is not** satisfied



Numeric locking (no sliding)

inf-sup **is** satisfied

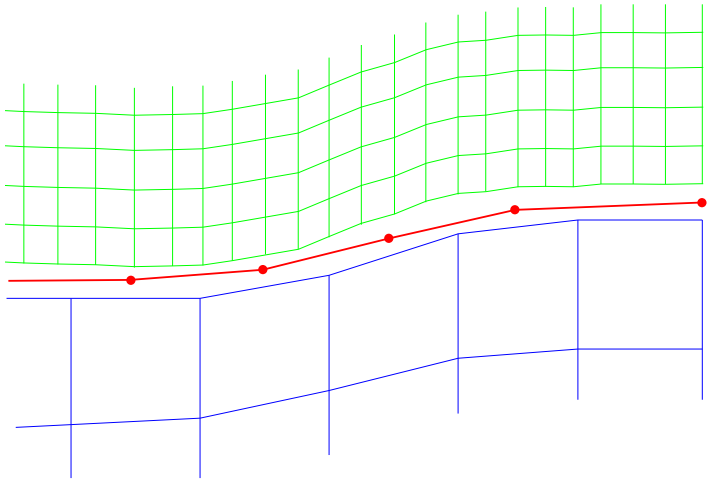


Interpenetration ($p_1 > p_0$)

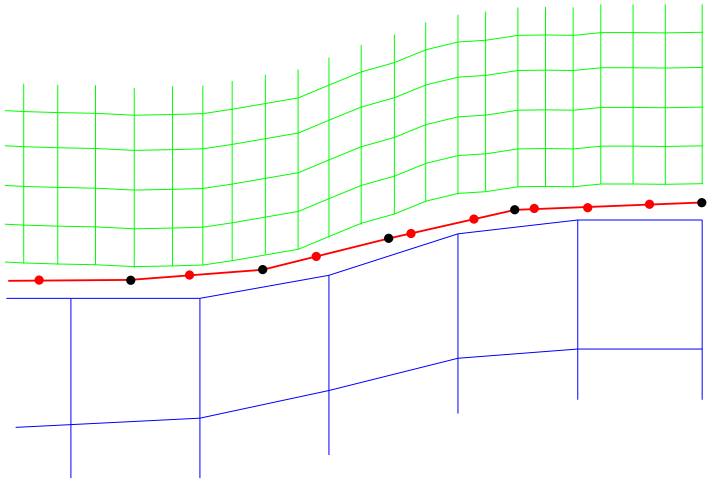
inf-sup **is** satisfied

Meshing method (two steps)

- Geometry



- Refinement



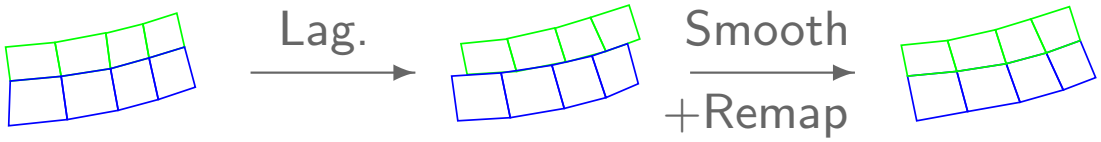
Stabilization and ALE

Stabilization

- Lagrangian stabilization by the way of sub-zonal entropy (sze): Després-Labourasse *J. Comput. Phys.*, 2012
  Linear system correction (AS_i: diagonal $d \times d$-bloc matrices)

$$\begin{pmatrix} A_1+AS_1 & 0 & {}^tB_1 \\ 0 & A_2+AS_2 & {}^tB_2 \\ B_1 & B_2 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \Lambda \end{pmatrix} = \begin{pmatrix} L_1+LS_1 \\ L_2+LS_2 \\ 0 \end{pmatrix}$$

Simple-ALE



- Lagrange+remap approach
- Initial global mesh is conform
- After lagrangian step, global mesh is no more conform
- Global mesh conformity is enforced while smoothing $\mathcal{M}_i$'s
- Remapping fluxes are canceled on $\Gamma \implies$ no mixing $\implies$ no mixing model

Outline

1 Abstract scheme without slide-lines

2 $\mathbb{P}_1$ discretization without slide-lines

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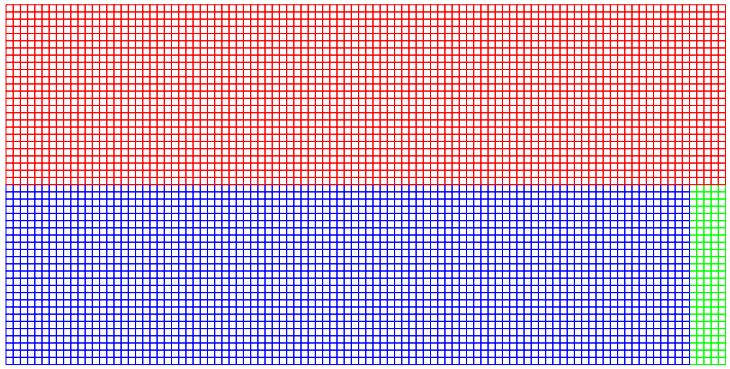
5 Numerical tests

6 Conclusions and perspectives

Caramana's test

Caramana *J. Comput. Phys.*, 2009

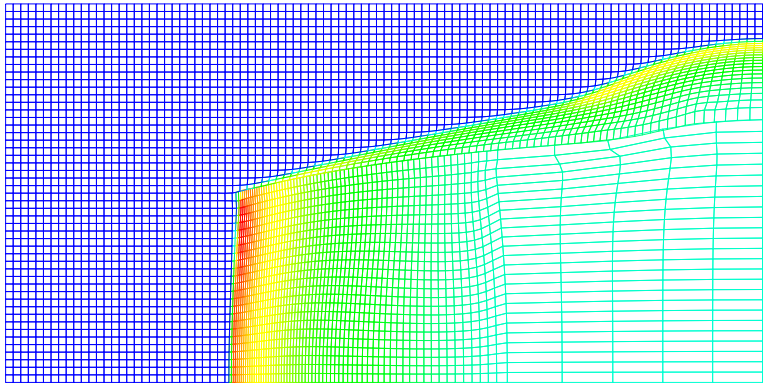


$\rho = 1$	$p = 20$	$(\epsilon = 30)$
$\rho = 1$	$p = \frac{2}{3} \times 10^{-8}$	$(\epsilon = 10^{-8})$
$\rho = 10$	$p = \frac{2}{3} \times 10^{-8}$	$(\epsilon = 10^{-9})$
$\mathbf{u} = \mathbf{0}$	$\gamma = \frac{5}{3}$	(perfect gas)

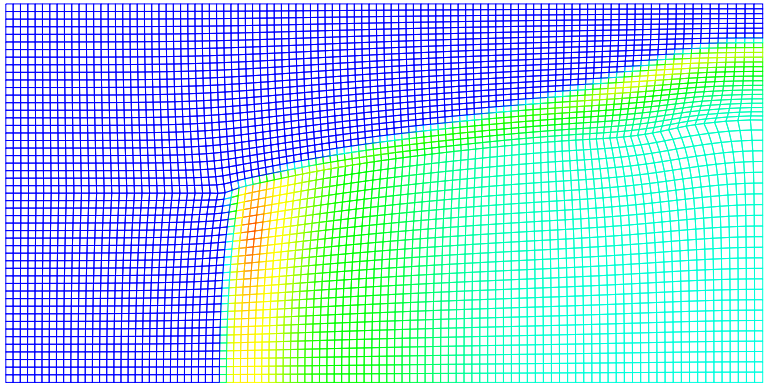
- $\Omega =]0, 1[\times]0.25, 0.5[\cup]0, 0.95[\times]0, 0.25[\cup]0.95, 1[\times]0, 0.25[$
- Symmetry boundary conditions all over $\partial\Omega$
- Final time: 0.3
- 2 meshes: 100×50 and 200×100
- Lagrangian calculations (sze: 0.1 on Γ_i 0.01 elsewhere)
- ALE (no sze) geometric smoothing, limited second-order remapping
- Away from slide-line solver: first-order Eucclhyd

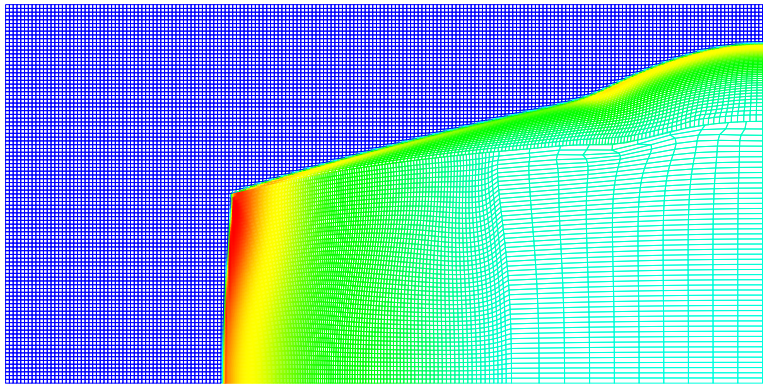
Caramana's test

Lagrangian | ALE

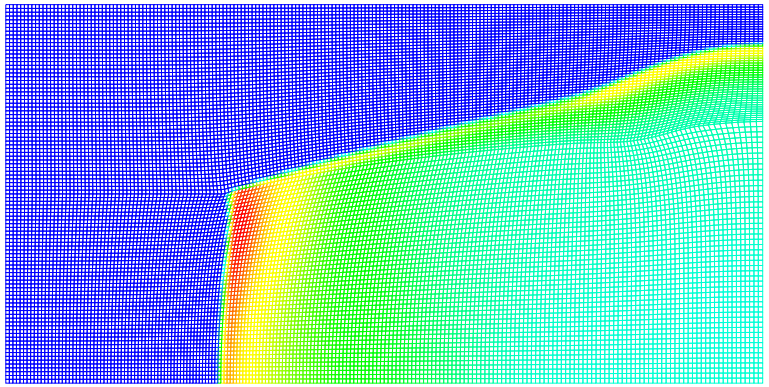


100×50





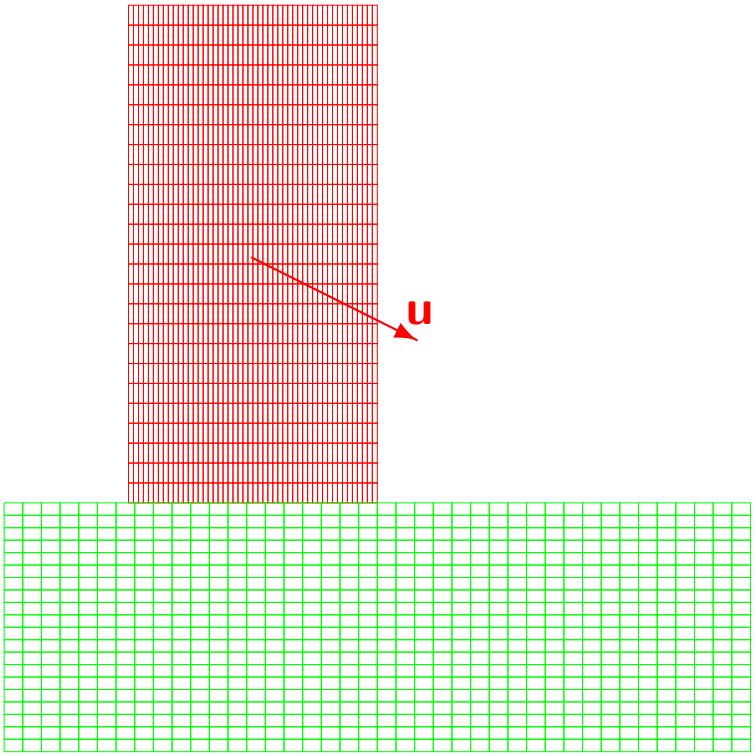
200×100



$t = 0.3$, pressure $\left[\frac{2}{3} \times 10^{-9}, 1.8\right]$

[animation]

Obstacle

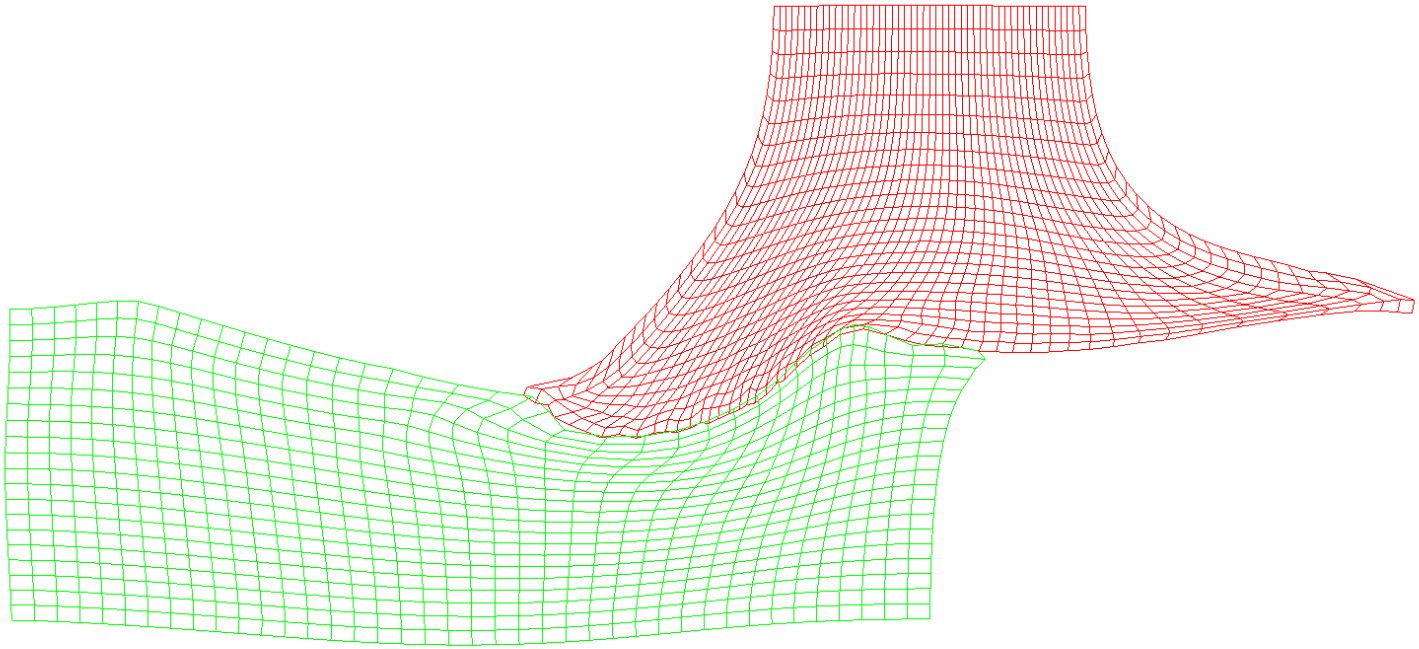


$\rho = 1 \qquad p = 0 \qquad \mathbf{u} = (2, -1)$

$\rho = 2 \qquad p = 0 \qquad \mathbf{u} = \mathbf{0}$

- $\Omega =]-\frac{1}{2}, 0[\times]1, \frac{1}{2}[\cup]-\frac{1}{4}, \frac{1}{2}[\times]\frac{1}{4}, \frac{3}{2}[$
- Final time: $t = \frac{1}{2}$
- $p(\rho, \epsilon) = (\gamma - 1)\rho\epsilon - \gamma p_0$
- $c = \sqrt{\gamma \frac{p+p_0}{\rho}}$
- $\gamma = \frac{5}{3}$ and $p_0 = 1$
- Away from slide-line Solver : Glace (ordre 1)
- sze: 0.01 at interface, 0.001 elsewhere
- Mesh: 50×25, 40×20

Obstacle (2/2)



Geometry at time $t = \frac{1}{2}$ [animation]

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Conclusions and perspectives

Conclusions

- Abstract mortar-like sliding method
  - Conservative in mass, momentum and total energy
  - Entropy production error is first-order in space
- $\mathbb{P}_1 - \mathbb{P}_0$ discretization of the saddle point problem $\implies (\mathbf{u}^\star_{1h}, \mathbf{u}^\star_{2h}, \lambda_h)$
  - Well-posed (inf-sup) if Γ_h locally coarser than the finest of $\Gamma'_{i,h}s$
  - $(\mathbf{u}^\star_{1h}, \mathbf{u}^\star_{2h}, \lambda_h)$ stable with regard to data
  - Requires the resolution of a linear system that couples all dof of $\Gamma_{i,h}$
  - Exact resolution to ensure conservation
- Solution quality depends on Γ's mesh
- Compatible with Simple ALE methods
- Numerical results demonstrate the validity of the approach

Conclusions and perspectives

Perspectives

- Improve Γ_h's generation (robustness, precision, entropy?,...)
- Other discrete space choices ($\mathbb{P}_1 - \mathbb{P}_1$?,... , curvilinear cells?, ...)
- Adapt method to the elasto-plastic case (should be straightforward)
  - Kluth-Després *J. Comput. Phys.* 2010
  - Maire, Abgrall, Breil, Loubère, Rebourcet *J. Comput. Phys.* 2013
  - Burton, Carney, Morgan, Sambasivan, Shashkov *Comput. Fluid.* 2013
  - ...
- Second-order accuracy, friction, impact
- Cylindrical geometry, 3D, parallelism,...
- Linear system resolution strategies (solvers, preconditioners,...)

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