DE LA RECHERCHE À L'INDUSTRIE

Contact algorithms for cell-centered lagrangian schemes

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September, the 3^{rd}, 2013

complexity made simple

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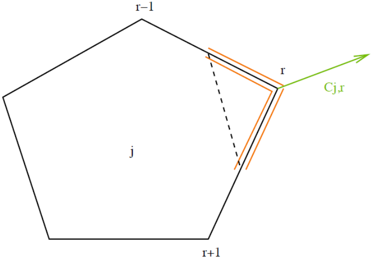
- We design contact algorithms for cell-centered lagrangian schemes to solve most of the problems for which different constraints must be taken into account
- In this presentation, we will focus on impact and pure sliding at interfacial boundaries between non-mixing media of solid, liquid or gas nature.
- Methods different from the usual *master/slave explicit* approach based on Wilkins' original work in staggered schemes.
- Cell-centered lagrangian schemes (GLACE Després, EUCCLHYD Maire) are usually based on a specific nodal solver to compute the node's velocity in order to move the mesh. We replace such nodal solver by a new elegant method of minimization under constraints.

Outline

The GLACE scheme
Overview for conformal meshes

Semi-discrete numerical approximation of Euler's equations :

$$\begin{cases} V_j' = (\nabla_{\mathbf{x}} V_j \cdot \mathbf{x}') = \displaystyle\sum_{r\in\mathcal{N}(j)} (\mathbf{C}_{j,r}\cdot\mathbf{u_r}) \\ M_j \tau_j'(t) = \displaystyle\sum_{r\in\mathcal{N}(j)} (\mathbf{C}_{j,r}\cdot\mathbf{u}_r) \\ M_j \mathbf{u}_j'(t) = -\displaystyle\sum_{r\in\mathcal{N}(j)} \mathbf{C}_{j,r} p_{j,r} \\ M_j e_j'(t) = -\displaystyle\sum_{r\in\mathcal{N}(j)} (\mathbf{C}_{j,r}\cdot\mathbf{u}_r) p_{j,r} \end{cases}$$



where M_j mass, V_j volume, τ_j specific volume, $\mathbf{u}$ velocity and e total energy. $\mathbf{C}_{j,r}$ are the corner vectors.

$p_{j,r}$ is the nodal pressure, based on a Riemann invariant formulation :

$$p_{j,r} - p_j + \alpha_j((\mathbf{u}_r - \mathbf{u}_j)\cdot\mathbf{n}_{j,r}) = 0,$$

where $\mathbf{n}_{j,r} = \dfrac{\mathbf{C}_{j,r}}{|\mathbf{C}_{j,r}|}$, $\alpha_j = \rho_j\, c_j$ the acoustic impedance and $\mathbf{u}_r$ the velocity of the r-th node.

The Riemann solver : $\boxed{\forall r \in [1:N], \sum_{j \in \mathcal{C}(r)} \mathbf{C}_{j,r} p_{j,r} = 0 \Longleftrightarrow A_r \mathbf{u}_r = \mathbf{b}_r}$

with

$$A_r = \sum_{j \mathcal{B}_r} \alpha_j \left(\mathbf{n}_{j,r} \otimes \mathbf{C}_{j,r} \right)$$

$$\mathbf{b}_r = \sum_{j \in \mathcal{B}_r} \mathbf{C}_{j,r} p_j + \sum_{j \mathcal{B}_r} \alpha_j \left(\mathbf{n}_{j,r} \otimes \mathbf{C}_{j,r} \right) \mathbf{u}_j$$

The unique solution is

$$\boxed{\forall r \in [1:N], \qquad \mathbf{u}_r = A_r^{-1} \mathbf{b}_r}$$

since A_r is a symmetric positive-definite matrix

Nodal velocities are solved *independently*

Solving $\forall r \in [1:N]\,,\ A_r\ \mathbf{u}_r = \mathbf{b}_r$ is equivalent to minimize within the set $\mathbb{R}^d$ the following objective function :

$$J_r : \mathbb{R}^d \to \mathbb{R}$$
$$\mathbf{u}_r \to J_r(\mathbf{u}_r) = \frac{1}{2}\left(A_r\mathbf{u}_r,\ \mathbf{u}_r\right) - \left(\mathbf{b}_r,\ \mathbf{u}_r\right)$$

We shall consider here $d = 1, 2$. Because constraints globally apply on the mesh, it is natural to define a global objective function $J(\mathbf{U}) = \sum_r J_r(\mathbf{u}_r)$ where $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N)^t$.

It can also be written in a more general form :

$$J : \mathbb{K} \to \mathbb{R}$$
$$\mathbf{U} \to J(\mathbf{U}) = \frac{1}{2}\left(A\mathbf{U},\ \mathbf{U}\right) - \left(\mathbf{B},\ \mathbf{U}\right)$$

where

$$A = \begin{pmatrix} A_1 & & \dots & & 0 \\ & \ddots & & & \\ \vdots & & A_r & & \vdots \\ & & & \ddots & \\ 0 & & \dots & & A_N \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_r \\ \vdots \\ \mathbf{b}_N \end{pmatrix}$$

- <u>unconstrained case</u> $\mathbb{K} = \mathbb{R}^{d\times N}$, $\mathbf{U}_{\text{min}} = A^{-1}\mathbf{B}$
- <u>constrained case</u> $\mathbb{K} = \{\mathbf{U} \in \mathbb{R}^{d\times N},\ \mathbf{F}(\mathbf{U}) \leq 0\}$

where $\mathbf{F} = (F_1(\mathbf{U}), \ldots, F_M(\mathbf{U}))^T$ are real functions expressing M constraints applying on all constrained nodes.

$$\begin{cases} A\mathbf{U}_{\text{min}} - \mathbf{B} + \boldsymbol{\lambda} F'(\mathbf{U}) = 0 \\ \mathbf{F}(\mathbf{U}) \leq 0 \end{cases}$$

$\boldsymbol{\lambda}$ lagrange multipliers

Properties

① $\mathbb{K}$ is non empty $\left(\mathbf{0} = (0, \ldots, 0)^t \in \mathbb{K}\right)$.

② $\mathbb{K}$ is closed.

③ $\mathbb{K}$ is convex.

④ If translations $\mathbf{W}_a = (\mathbf{a}, \ldots, \mathbf{a})^t$ $(\mathbf{a} \in \mathbb{R}^d)$ are elements of $\mathbb{K}$, then *momentum is preserved.*

⑤ if $\exists \mu > 0, (1-\mu)\,\mathbf{U} \in \mathbb{K}$ and $(1+\mu)\,\mathbf{U} \in \mathbb{K}$, then *total energy is preserved.*

Rem : the most convenient case is that $\mathbb{K}$ is a cone : $\forall \mathbf{U} \in \mathbb{K}, \forall \lambda > 0, \lambda\mathbf{U} \in \mathbb{K}$

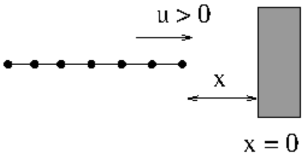
- No change in the CFL condition :

$$\max_j \left(\frac{c_j}{\Delta x_j} \right) \Delta t \leq 1$$

where we choose $\Delta x_j = \dfrac{V_j}{\sum_r \sqrt{\mathbf{C}_{j,r} \cdot \mathbf{C}_{j,r}}}$

- the 2nd law of thermodynamics is satisfied.

$\boxed{1D}$ Impact Problem of a mobile against a wall



$$\boxed{\forall t > 0,\ \forall x \in \Omega, \quad x(t) \leq 0}$$

Discrete :

$$\mathbb{K}_n = \{\mathbf{U} \in \mathbb{R}^N,\ \forall j \in [1:N],\ x^n_{j+1/2} + \Delta t u^n_{j+1/2} \leq 0\} = \mathbb{R}_-$$

Unconstrained solution : $\quad u^{\text{uncons}}_{N+1/2} = u_N + \frac{P_N - P_{\text{ext}}}{\rho_N c_N}$

- if $u_{N+1/2} < \frac{-x^n_{j+1/2}}{\Delta t}$ (inactive constraint), then $u_{N+1/2} = u^{\text{uncons}}_{N+1/2}$
- if $u_{N+1/2} \geq \frac{-x^n_{j+1/2}}{\Delta t}$, then $u_{N+1/2} = -\frac{x^n_{j+1/2}}{\Delta t}$

Properties of $\mathbb{K}_n$:

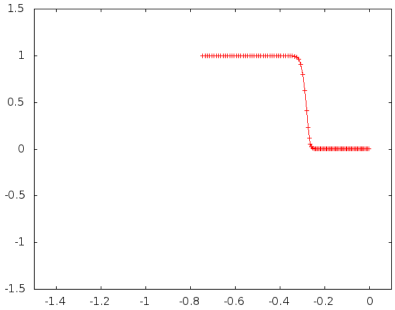
① $\mathbb{K}_n$ is non empty.

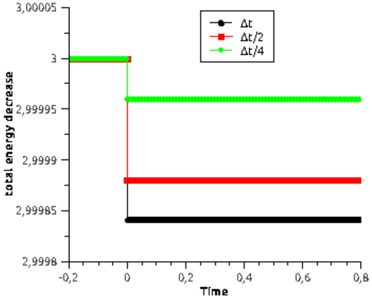
② $\mathbb{K}_n$ is closed.

③ $\mathbb{K}_n$ is convex.

④ *Preservation of momentum* As long as impact occurs, $u_{N+1/2} = 0$: there are no translations $\mathbf{W}_a$ with $a > 0$. Momentum then changes (it decreases).

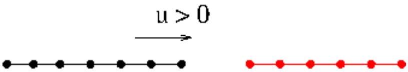
⑤ *Preservation of total energy* Over the timestep of impact, $u_{N+1/2} = -\frac{x^n_{N+1/2}}{\Delta t}$: $\nexists \mu > 0, (1+\mu)\, u_{N+1/2} \in \mathbb{K}_n$. Total energy changes (it decreases). This decrease is $\mathcal{O}(\Delta t)$.





Numerical examples
1D framework

$\boxed{1D}$ Impact Problem between two mobiles



$$\boxed{\forall t > 0, \forall (x,y) \in \Omega_1 \times \Omega_2, x(t) < y(t)}$$

$$\mathbb{K}_n = \{\mathbf{U} \in \mathbb{R}^N,\ \forall (j,k) \in [1:N] \times [N+1:2N],\ x^n_{j+1/2} + \Delta t u^n_{j+1/2} \leq x^n_{k+1/2} + \Delta t u^n_{k+1/2}\}$$

Constraint may be reformulated as :

$$u^n_{Nj+1/2} - u^n_{Nk+1/2} \leq \frac{x^n_{k+1/2} - x^n_{j+1/2}}{\Delta t} \ (= u_{\text{lim}})$$

Unconstrained solutions :

$$u^{\text{uncons}}_{Nj+1/2} = u_N + \frac{P_N - P_{\text{ext}}}{\rho_N c_N}$$

$$u^{\text{uncons}}_{Nk+1/2} = u_{N+1} + \frac{P_{\text{ext}} - P_{N+1}}{\rho_{N+1} c_{N+1}}$$

Numerical examples
1D framework

Properties of $\mathbb{K}_n$:

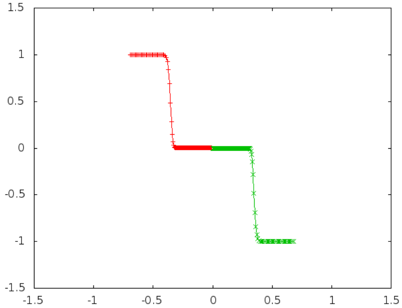
① $\mathbb{K}_n$ is non empty.

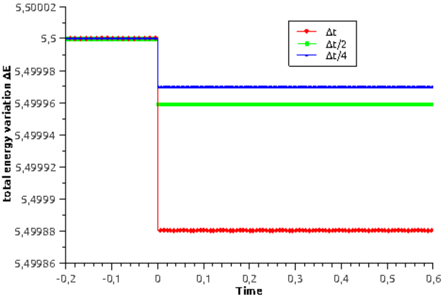
② $\mathbb{K}_n$ is closed.

③ $\mathbb{K}_n$ is convex.

④ *Preservation of momentum* All the translations are admissible. Momentum is preserved.

⑤ *Preservation of total energy* At the time of impact, $\left(u^n_{Nj+1/2} - u^n_{Nk+1/2}\right) = u_{\mathrm{lim}} : \nexists \mu > 0, (1+\mu)\,\mathbf{U} \in \mathbb{K}_n \implies$ total energy changes (it decreases). This decrease is $\mathcal{O}(\Delta t)$.

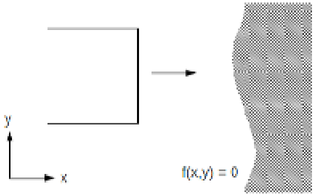




Additional Remarks

1. In general, the set $\mathbb{K}_n$ may be written for the set of nodes within the mesh which constraints apply on. Others nodal velocities may be computed with the usual Riemann Solvers $\rightarrow$ calculation time
2. We use a single mesh
3. We don't need to compute the time of impact and thus to treat two distinct cases (for $t \leq t_c$ and $t > t_c$).

$\boxed{2D}$ Impact Problem of mobile against a wall



$$\boxed{\forall t > 0,\ \forall \mathbf{x} \in \Omega, f(\mathbf{x}(t)) \leq 0}$$

$\mathbf{x}_r = (x_r^n, y_r^n)^t \qquad \mathbf{u}_r^n = (u_r^n, v_r^n)^t$

$$\mathbb{K}_n = \{\mathbf{U} \in \mathbb{R}^N,\ \forall r \in [1:N], f(\mathbf{x}_r^{n+1}) \leq 0\}$$

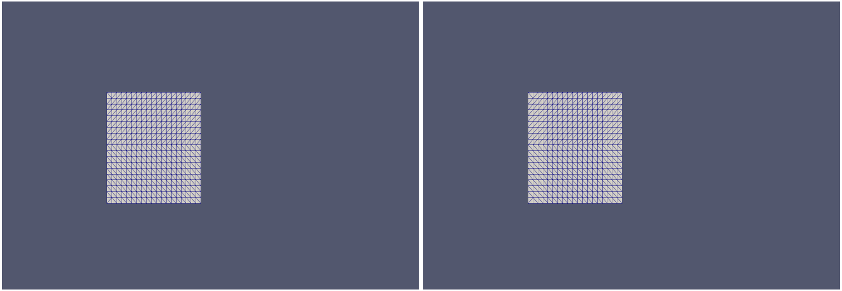
case of a plane wall $(x = 0)$

$$\forall r \in [1:N], x_r^n + \Delta t u_r^n \leq 0$$

case of a concave wall ($x + y^2 = 0$)

$$\forall r \in [1:N], x_r^n + \Delta t u_r^n + (y_r^n + \Delta t v_r^n)^2 \leq 0$$

Rem : Constraints are all independent. We can solve N independent constrained problems using the J_r functions.



Properties of $\mathbb{K}$:

① $\mathbb{K}$ is non empty.

② $\mathbb{K}$ is closed.

③ $\mathbb{K}$ is convex.

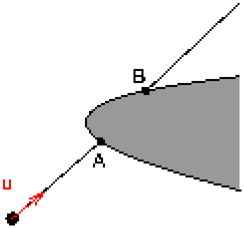
④ *Preservation of momentum* As long as impact occurs, $(\mathbf{u}_r^n.\mathbf{n}) = 0$ ($\mathbf{n}$ the outward pointing normal of the wall) : there are no translations $\mathbf{W}_a$ with $(\mathbf{a} \cdot \mathbf{n}) < 0$. Momentum then changes (it decreases).

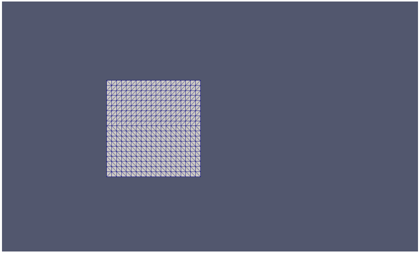
⑤ *Preservation of total energy* Property ④ is violated each time a node impacts the wall. Total energy decreases as a consequence, and each decreases is $\mathcal{O}(\Delta t)$.

case of a convex wall $(x + y^2 = 0)$

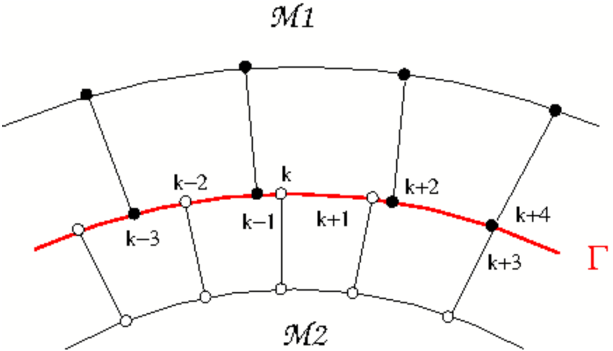
$$\forall r \in [1 : N], x_r^n + \Delta t u_r^n - (y_r^n + \Delta t v_r^n)^2 \leq 0$$

Rem : $\mathbb{K}$ is concave ! Property ③ is not satisfied and the solution might be non unique $\Longrightarrow$ The good solution is captured by reducing the timestep.





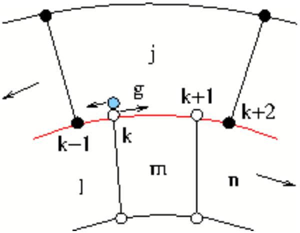
$\boxed{2D}$ Sliding between two fluids



$$\boxed{\forall t > 0, \forall \mathbf{x} \in \Gamma, \left(\mathbf{u}(\mathbf{x}(t), t)^{+} - \mathbf{u}(\mathbf{x}(t), t)^{-},\ \mathbf{n}(\mathbf{x}(t))\right) = 0}$$

Numerical examples
2D Framework - Sliding

$$\mathbb{K}_n = \left\{ \mathbf{U} \in \mathbb{R}^N,\ \forall k \in [1:N]\,, \left(\mathbf{u}_k^{n,+} - \mathbf{u}_k^{n,-}, \mathbf{n}_{\mathrm{RES}} \right) = 0 \right\}$$



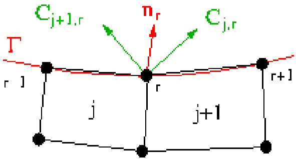
$$\mathbf{u}_k^{n,-} = \mathbf{u}_k$$

$$\mathbf{u}_k^{n,+} = \mathbf{u}_k^{n,g} = P_1^k\ \mathbf{u}_{k-1}^n + P_2^k\ \mathbf{u}_{k+2}^n$$

$$P_1^k \text{ and } P_2^k \in \mathbb{R}^{2\times 2}$$

$$\mathbf{n}_{\mathrm{RES}} = \frac{1}{2}\left(\mathbf{n}_k^n + \mathbf{n}_k^{n,g}\right)$$

$$\mathbf{n}_k^{n,g} = \frac{\left(\mathbf{x}_{k-1}\mathbf{x}_{k+2}\right)^{\perp}}{\left|\mathbf{x}_{k-1}\mathbf{x}_{k+2}\right|^{\perp}} \qquad \mathbf{n}_k^n = \frac{\sum_j \mathbf{C}_{j,r}}{\left|\sum_j \mathbf{C}_{j,r}\right|}$$



$$\mathbb{K}_n = \{\mathbf{U} \in \mathbb{R}^N,\ \forall k \in [1:N]\,, \left(\mathbf{u}_k^{n,+} - \mathbf{u}_k^{n,-}, \mathbf{n}_{\mathsf{RES}}\right) = 0\}$$

Rem : All the constrained node are now coupled. The use of the global function J is required.

Properties of $\mathbb{K}_n$:

① $\mathbb{K}_n$ is non empty.

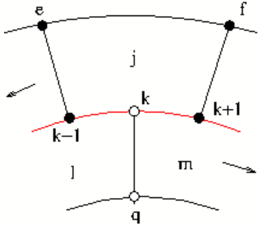
② $\mathbb{K}_n$ is closed.

③ $\mathbb{K}_n$ is convex.

④ *Preservation of momentum* Translations are admissible. Momentum is preserved.

⑤ *Preservation of total energy* $\mathbb{K}_n$ is a cone. Total Energy is preserved.

Computation of the cell volume :



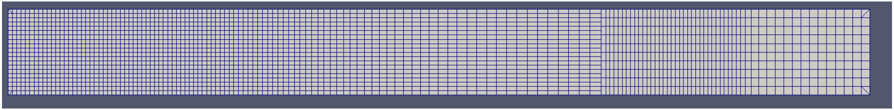
The volume of the cell 'j' obviously depends on **k** :

$$
\begin{aligned}
V_j &= \frac{1}{d} \sum_{r \in \mathcal{B}_j} \left(\mathbf{C}_{j,r} \cdot \mathbf{x}_r \right) \\
&= \frac{1}{2} \left(\mathbf{C}_{j,e} \cdot \mathbf{x}_e + \mathbf{C}_{j,f} \cdot \mathbf{x}_f + \mathbf{C}_{j,k-1} \cdot \mathbf{x}_{k-1} + \mathbf{C}_{j,k+1} \cdot \mathbf{x}_{k+1} + \mathbf{C}_{j,k} \cdot \mathbf{x}_k \right)
\end{aligned}
$$

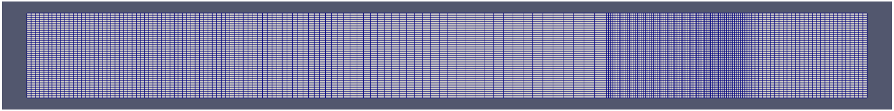
- The full volume is preserved.
- No void is created

Sod test Case with an initial artificial slide line :

Sliding line coincident to the initial discontinuity



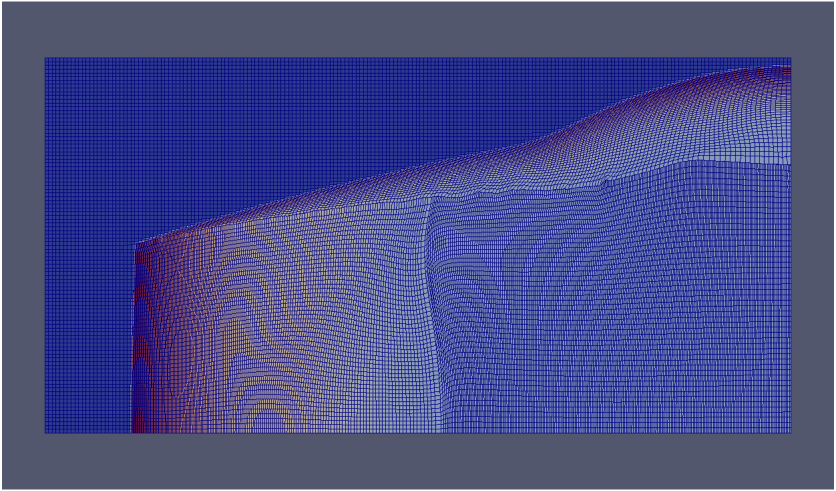
Sliding line parallel to the flow



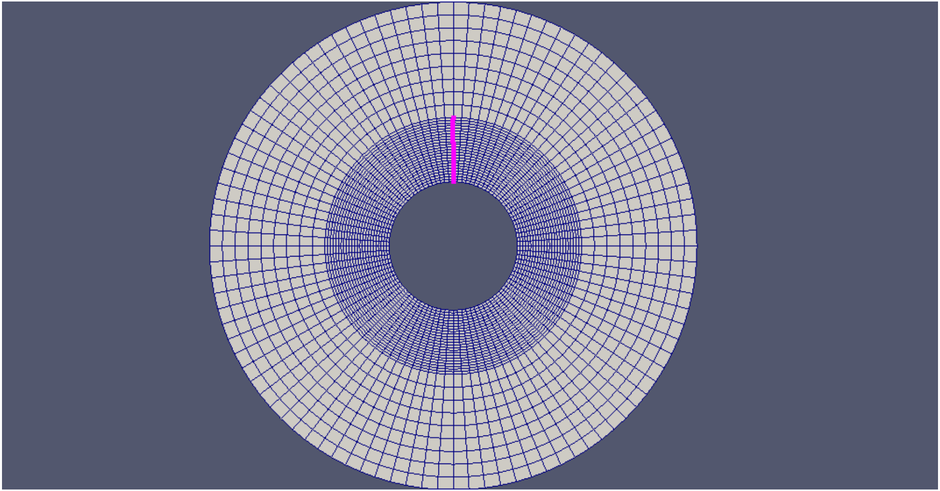
In both cases :

- the position of the sliding line, as well as the symetry of the problem, are preserved in both case.
- The convergence of the numerical solution is ensured.
- In this case, momentum and total energy are conserved up to $\epsilon = 10^{-17}$ (Machine epsilon)

Caramana test Case :



Sliding Rings test Case :



Conclusions and perspectives

- Our method is based on the reformulation of the usual Riemann solvers used to compute the nodal velocity.
- An objective function is minimized within a set of admissible velocities $\mathbb{K}_n$.
- Providing that $\mathbb{K}_n$ satisfies the conditions ①-⑤, the method preserves mass, volume, momentum and total energy (up to the precision ϵ given in the minimization procedure)
- Our numerical test cases prove that the method is easy to implement and robust.
- The method can be extended to the 3D Framework, providing a new relation for the ghost node velocity

$$\mathbf{u}_k^{n,g} = Q_1^k\ \mathbf{u}_{k-1} + Q_2^k\ \mathbf{u}_{k+1} + Q_3^k\ \mathbf{u}_{k-2} + Q_4^k\ \mathbf{u}_{k+2}$$

- Several instabilities can be prevented by using the method of stabilization of B. Després and E. Labourasse (Després, JCP 2012)
- Adding several physical phenomena like friction and surface tension at the interface boundaries.

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