High Order Arbitrary-Lagrangian-Eulerian One-Step WENO Finite Volume Schemes on Unstructured Meshes

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Outline

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   - High Order WENO Reconstruction on Unstructured Meshes
   - Local Space-Time CG Predictor on Moving Meshes
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   - Solution Algorithm
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   - The Kidder Problem
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   - The ideal MHD equations
   - The MHD Blast Problem
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3. Conclusions and Outlook

Hyperbolic Balance Laws

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{Q}) = \mathbf{S}(\mathbf{Q}), \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^2, t \in \mathbb{R}_0^+$$

where

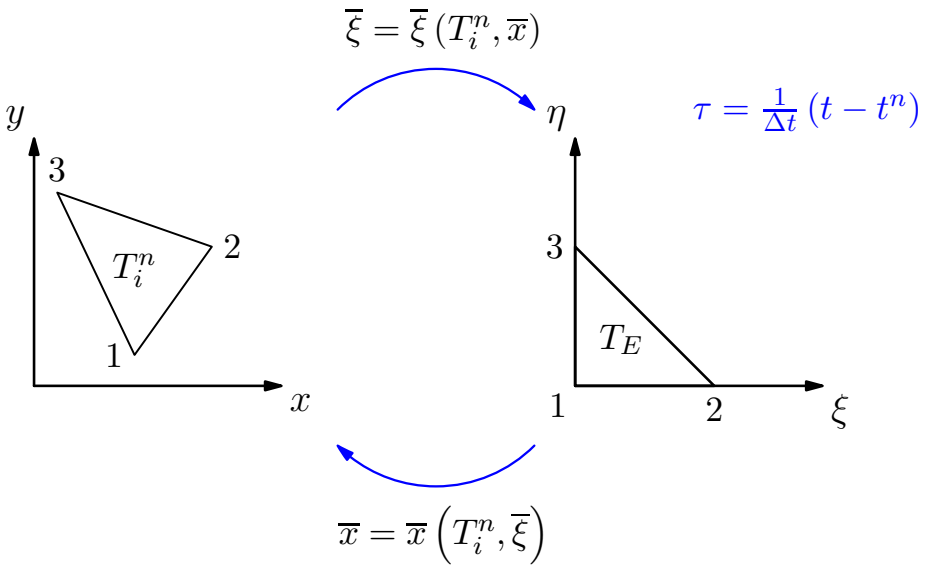
$\mathbf{Q} \in \Omega_Q \subset \mathbb{R}^\nu$	$\rightarrow$	state vector
$\mathbf{F} = (\mathbf{f}, \mathbf{g})$	$\rightarrow$	flux tensor
$\mathbf{S}(\mathbf{Q})$	$\rightarrow$	vector of algebraic source terms

Hyperbolic Balance Laws

Moving mesh

$$\mathcal{T}_\Omega^n = \bigcup_{i=1}^{N_E} T_i^n$$

$$\begin{aligned} x &= X_{1,i}^n + \left(X_{2,i}^n - X_{1,i}^n\right)\xi + \left(X_{3,i}^n - X_{1,i}^n\right)\eta \\ y &= Y_{1,i}^n + \left(Y_{2,i}^n - Y_{1,i}^n\right)\xi + \left(Y_{3,i}^n - Y_{1,i}^n\right)\eta \end{aligned}$$



Data representation

$$\mathbf{Q}_i^n = \frac{1}{|T_i^n|}\int_{T_i^n} \mathbf{Q}(x, y, t^n)dV$$

High Order Accuracy in Space

Some spatial basis functions $\psi_l(\xi,\eta)$ are used to write the reconstructed polynomials $\mathbf{w}_h^s$:

$$\mathbf{w}_h^s(x,y,t^n) = \sum_{l=1}^{\mathcal{M}} \psi_l(\xi,\eta)\hat{\mathbf{w}}_{l,i}^{n,s} := \psi_l(\xi,\eta)\hat{\mathbf{w}}_{l,i}^{n,s}$$

Integral conservation

$$\frac{1}{|T_j^n|}\int_{T_j^n} \psi_l(\xi,\eta)\hat{\mathbf{w}}_{l,i}^{n,s}\,dV = \mathbf{Q}_j^n, \qquad \forall T_j^n \in \mathcal{S}_i^s$$

$$\begin{aligned} \mathcal{S}_i^s &= \bigcup_{j=1}^{n_e} T_{m(j)}^n \\ \mathcal{M} &= (M+1)(M+2)/2 \\ n_e &= 2\mathcal{M} \end{aligned}$$

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WENO polynomials

$$\mathbf{w}_h(x,y,t^n) = \sum_{l=1}^{\mathcal{M}} \psi_l(\xi,\eta)\hat{\mathbf{w}}_{l,i}^n, \qquad \text{with} \qquad \hat{\mathbf{w}}_{l,i}^n = \sum_s \omega_s \hat{\mathbf{w}}_{l,i}^{n,s}$$

nonlinear WENO weights

$$\tilde{\omega}_s = \frac{\lambda_s}{(\sigma_s+\epsilon)^r}, \quad \omega_s = \frac{\tilde{\omega}_s}{\sum_q \tilde{\omega}_q}$$

oscillation indicators

$$\sigma_s = \left(\sum_{\alpha+\beta\le M} \int_{T_e} \frac{\partial^{\alpha+\beta}\psi_l(\xi,\eta)}{\partial\xi^\alpha\partial\eta^\beta} \cdot \frac{\partial^{\alpha+\beta}\psi_m(\xi,\eta)}{\partial\xi^\alpha\partial\eta^\beta} d\xi d\eta \right) \cdot \hat{w}_{l,i}^{n,s}\hat{w}_{m,i}^{n,s}$$

High Order Accuracy in Time

$$\tilde{\mathbf{x}} = (x, y, t) \qquad \tilde{\boldsymbol{\xi}} = (\xi, \eta, \tau) \qquad \theta_l = \theta_l(\tilde{\boldsymbol{\xi}}) = \theta_l(\xi, \eta, \tau)$$

$$\mathbf{x} = (x, y) \qquad \boldsymbol{\xi} = (\xi, \eta)$$

physical coordinates | reference coordinates | space–time basis functions

Isoparametric approach

$$\mathbf{q}_h = \mathbf{q}_h(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{\mathbf{q}}_{l,i} \qquad \mathbf{S}_h = \mathbf{S}_h(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{\mathbf{S}}_{l,i}$$

$$\mathbf{f}_h = \mathbf{f}_h(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{\mathbf{f}}_{l,i} \qquad \mathbf{g}_h = \mathbf{g}_h(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{\mathbf{g}}_{l,i}$$

$$\Downarrow$$

$$x(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{x}_{l,i}, \qquad y(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{y}_{l,i}, \qquad t(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau)\widehat{t}_l$$

Jacobian

$$J_{st} = \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{\boldsymbol{\xi}}} = \begin{pmatrix} x_\xi & x_\eta & x_\tau \\ y_\xi & y_\eta & y_\tau \\ 0 & 0 & \Delta_t \end{pmatrix} \qquad J_{st}^{-1} = \frac{\partial \tilde{\boldsymbol{\xi}}}{\partial \tilde{\mathbf{x}}} = \begin{pmatrix} \xi_x & \xi_y & \xi_t \\ \eta_x & \eta_y & \eta_t \\ 0 & 0 & \frac{1}{\Delta t} \end{pmatrix}$$

High Order Accuracy in Time

Weak formulation of the PDE

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \Delta t \left[\frac{\partial \mathbf{Q}}{\partial \xi}\xi_t + \frac{\partial \mathbf{Q}}{\partial \eta}\eta_t + \frac{\partial \mathbf{f}}{\partial \xi}\xi_x + \frac{\partial \mathbf{f}}{\partial \eta}\eta_x + \frac{\partial \mathbf{g}}{\partial \xi}\xi_y + \frac{\partial \mathbf{g}}{\partial \eta}\eta_y \right] = \Delta t \mathbf{S}(\mathbf{Q})$$

Let define

$$[f,g]^{\tau} = \int_{T_e} f(\xi,\eta,\tau) g(\xi,\eta,\tau) d\xi d\eta, \quad \langle f,g \rangle = \int_0^1 \int_{T_e} f(\xi,\eta,\tau) g(\xi,\eta,\tau) d\xi d\eta d\tau$$

High Order Accuracy in Time

Weak formulation of the PDE

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \Delta t \left[\frac{\partial \mathbf{Q}}{\partial \xi}\xi_t + \frac{\partial \mathbf{Q}}{\partial \eta}\eta_t + \frac{\partial \mathbf{f}}{\partial \xi}\xi_x + \frac{\partial \mathbf{f}}{\partial \eta}\eta_x + \frac{\partial \mathbf{g}}{\partial \xi}\xi_y + \frac{\partial \mathbf{g}}{\partial \eta}\eta_y \right] = \Delta t \mathbf{S}(\mathbf{Q})$$

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Integration over a moving space–time control volume $T_i^n \times \left[t^n, t^{n+1}\right]$ yields

$$\begin{aligned}
\left\langle \theta_k, \frac{\partial \theta_l}{\partial \tau} \right\rangle \widehat{\mathbf{q}}_h \quad &+ \quad \Delta t \left[\left\langle \theta_k, \frac{\partial \theta_l}{\partial \xi}\xi_t \right\rangle \widehat{\mathbf{q}}_h + \left\langle \theta_k, \frac{\partial \theta_l}{\partial \eta}\eta_t \right\rangle \widehat{\mathbf{q}}_h \right] \\
&+ \quad \Delta t \left[\left\langle \theta_k, \frac{\partial \theta_l}{\partial \xi}\xi_x \right\rangle \widehat{\mathbf{f}}_h + \left\langle \theta_k, \frac{\partial \theta_l}{\partial \eta}\eta_x \right\rangle \widehat{\mathbf{f}}_h \right] \\
&+ \quad \Delta t \left[\left\langle \theta_k, \frac{\partial \theta_l}{\partial \xi}\xi_y \right\rangle \widehat{\mathbf{g}}_h + \left\langle \theta_k, \frac{\partial \theta_l}{\partial \eta}\eta_y \right\rangle \widehat{\mathbf{g}}_h \right] = \langle \theta_k, \theta_l \rangle \widehat{\mathbf{S}}_h
\end{aligned}$$

High Order Accuracy in Time

$$K_\tau \widehat{\mathbf{q}}_{l,i} + \Delta t \left(K_t \widehat{\mathbf{q}}_{l,i} + K_x \widehat{\mathbf{f}}_{l,i} + K_y \widehat{\mathbf{g}}_{l,i} \right) = \Delta t M \widehat{\mathbf{S}}_{l,i}$$

Let define a unified term $\mathbf{P}$ as

$$\mathbf{P} := \mathbf{S}(\mathbf{Q}) - (\mathbf{Q}_\xi \xi_t + \mathbf{Q}_\eta \eta_t + \mathbf{f}_\xi \xi_x + \mathbf{f}_\eta \eta_x + \mathbf{g}_\xi \xi_y + \mathbf{g}_\eta \eta_y)$$

High Order Accuracy in Time

$$K_\tau \widehat{\mathbf{q}}_{l,i} + \Delta t \left(K_t \widehat{\mathbf{q}}_{l,i} + K_x \widehat{\mathbf{f}}_{l,i} + K_y \widehat{\mathbf{g}}_{l,i} \right) = \Delta t M \widehat{\mathbf{S}}_{l,i}$$

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$$\Downarrow$$

$$\mathbf{Q}_\tau = \Delta t \mathbf{P} \qquad \mathbf{P}_h = \mathbf{P}_h(\xi, \eta, \tau) = \theta_l(\xi, \eta, \tau) \widehat{\mathbf{P}}_{l,i}$$

$$\Downarrow$$

$$K_\tau \widehat{\mathbf{q}}_{l,i}^{r+1} = \Delta t M \widehat{\mathbf{P}}_{l,i}^{r}$$

Evolution of Vertex Coordinates

Local evolution:

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{Q}, \mathbf{x}, t), \qquad \mathbf{V}_h = \theta_l(\boldsymbol{\xi}, \tau)\widehat{\mathbf{V}}_{l,i}$$

$$\Downarrow$$

$$K_1 \widehat{\mathbf{x}}_{l,i} = [\theta_k(\boldsymbol{\xi}, 0), \mathbf{x}(\boldsymbol{\xi}, t^n)]^0 + \Delta t \mathbf{M}\, \widehat{\mathbf{V}}_{l,i}$$

Evolution of Vertex Coordinates

Local evolution:

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{Q}, \mathbf{x}, t), \qquad \mathbf{V}_h = \theta_l(\boldsymbol{\xi}, \tau)\widehat{\mathbf{V}}_{l,i}$$

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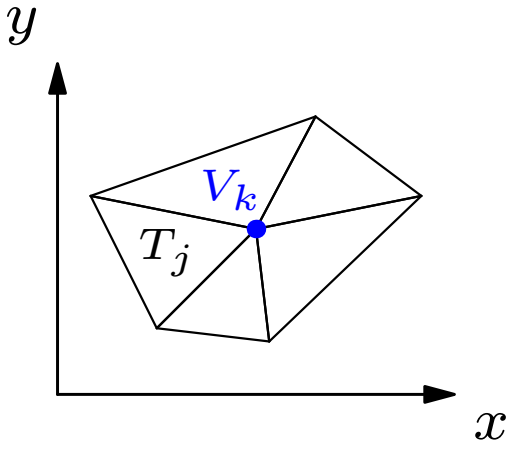
$$K_1 \widehat{\mathbf{x}}_{l,i} = [\theta_k(\boldsymbol{\xi}, 0), \mathbf{x}(\boldsymbol{\xi}, t^n)]^0 + \Delta t \mathbf{M}\, \widehat{\mathbf{V}}_{l,i}$$

Global evolution:

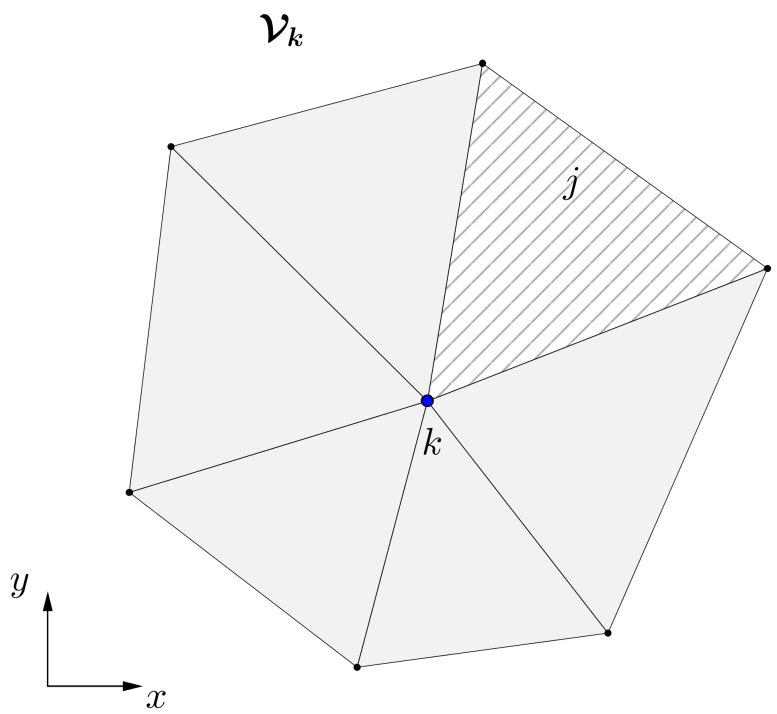


$\rightarrow$ *node solver*

Node Solvers

1) $\mathcal{NS}_{cs}$ (Cheng and Shu [3])

$$\overline{\mathbf{V}}_k^n = \frac{1}{N_k} \sum_{j \in \mathcal{V}_k} \overline{\mathbf{V}}_{k,j}^n \quad \text{with} \quad \overline{\mathbf{V}}_{k,j}^n = \left(\int_0^1 \theta_l(\xi_{e,m(k)}, \eta_{e,m(k)}, \tau) d\tau \right) \widehat{\mathbf{V}}_{l,j}$$



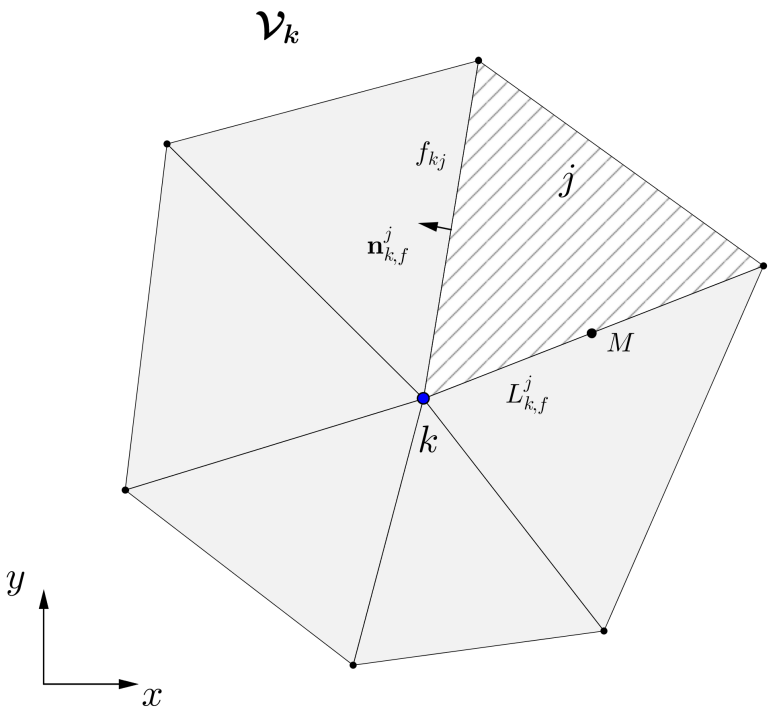
Node Solvers

2) $\mathcal{NS}_m$ (Maire [7])

$$\overline{\mathbf{V}}_k^n = M_k^{-1} \sum_{j \in \mathcal{V}_k} \left(L_{k,j} P_j \mathbf{n}_j + M_{p,j} \mathbf{V}_j \right)$$

with

$$M_{k,j} = \sum_{f \in j} Z_{k,f}^j L_{k,f}^j \left(\mathbf{n}_{k,f}^j \otimes \mathbf{n}_{k,f}^j \right), \quad M_k = \sum_{j \in \mathcal{V}_k} M_{k,j}, \quad Z_{k,f}^j = \rho_j c_j$$

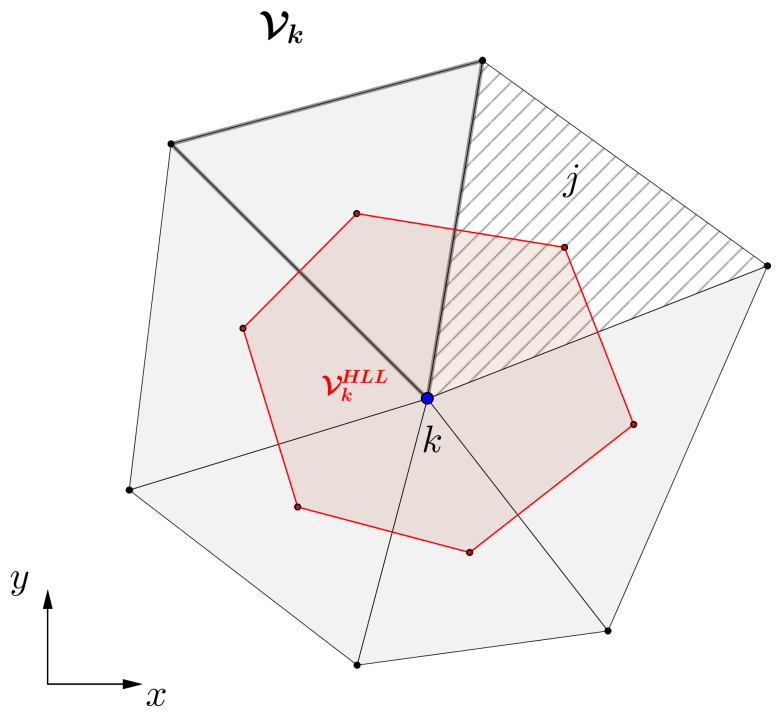


Node Solvers

3) $\mathcal{NS}_b$ (Balsara et al. [1])

$$\overline{\mathbf{V}}_k^n = \overline{\mathbf{V}}_k^{HLL}$$

It resolves the multidimensional Riemann problem in the neighborhood of vertex k.



Rezoning

- *Lagrangian coordinates*

$$\mathbf{x}_k^{n+1,Lag} = \mathbf{x}_k^n + \overline{\mathbf{V}}_k^n \cdot \Delta t$$

- *Rezoned coordinates*

$$F_k(\mathbf{x}_k) = \sum_{j \in \mathcal{V}_k} \mathcal{K}(\mathbf{J}_j), \quad \mathbf{J}_j = \begin{pmatrix} x_{k+1}^j - x_k^j & x_{k-1}^j - x_k^j \\ y_{k+1}^j - y_k^j & y_{k-1}^j - y_k^j \end{pmatrix}.$$

$$\mathbf{x}_k^{rez} \quad \rightarrow \quad min\left\{F_k(\mathbf{x}_k)\right\}$$

- *New coordinates* (relaxation algorithm [1])

$$\mathbf{x}_k^{n+1} = \mathbf{x}_k^{n+1,Lag} + \omega_k \left(\mathbf{x}_k^{rez} - \mathbf{x}_k^{n+1,Lag}\right), \qquad \omega_k = \mathcal{F}\left(\mathbf{x}_k^n, \mathbf{x}_k^{n+1,Lag}\right)$$

[1] Galera et al., *JCP* 2010

Finite Volume Scheme

The starting PDE system can be rewritten as

$$\nabla_{3D} \cdot \tilde{\mathbf{Q}} = \mathbf{S}(\mathbf{Q})$$

with

$$\tilde{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t} \right)^T, \qquad \tilde{\mathbf{Q}} = (\mathbf{F}, \mathbf{Q}) = (\mathbf{f}, \mathbf{g}, \mathbf{Q})$$

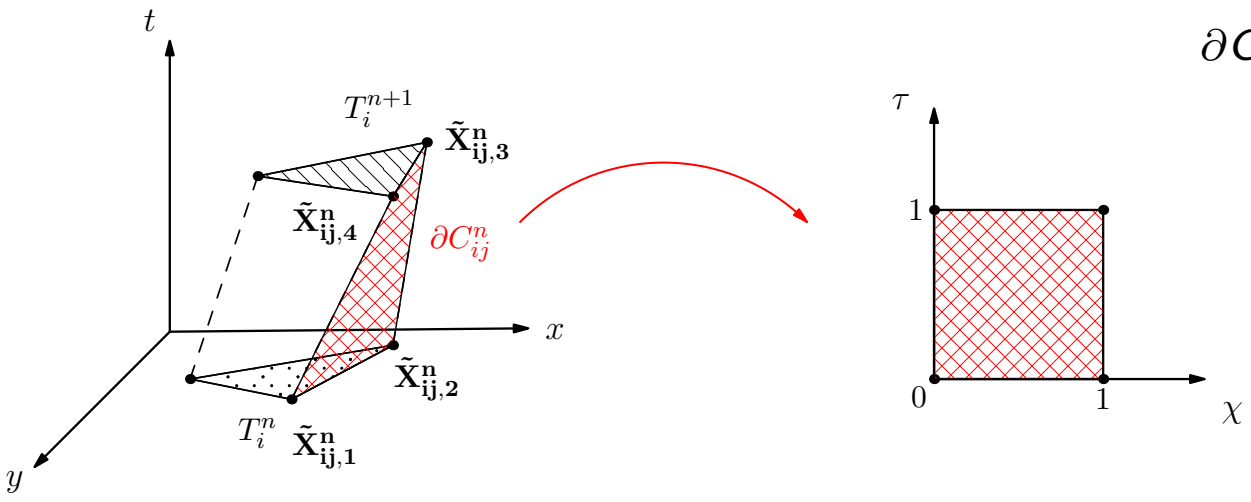
Integration over the space–time control volume $C_i^n = T_i(t) \times \left[t^n; t^{n+1}\right]$ yields

$$\int_{t^n}^{t^{n+1}} \int_{T_i(t)} \tilde{\nabla} \cdot \tilde{\mathbf{Q}} \, d\mathbf{x}dt = \int_{t^n}^{t^{n+1}} \int_{T_i(t)} \mathbf{S} \, d\mathbf{x}dt$$

$$\Downarrow$$

$$\int_{\partial C_i^n} \tilde{\mathbf{Q}} \cdot \tilde{\mathbf{n}} \, dS = \int_{t^n}^{t^{n+1}} \int_{T_i(t)} \mathbf{S} \, d\mathbf{x}dt$$

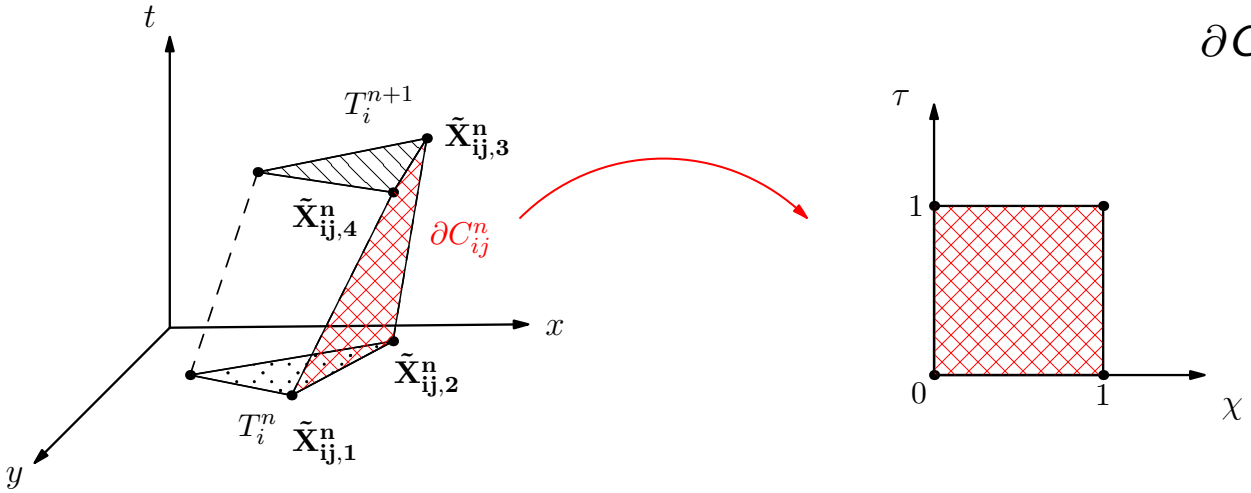
Finite Volume Scheme



$$\partial C_{ij}^n = \tilde{\mathbf{x}}(\chi,\tau) = \sum_{k=1}^{4} \beta_k(\chi,\tau)\tilde{\mathbf{X}}_{ij,k}^n$$

$$0 \le \chi \le 1, \quad 0 \le \tau \le 1$$

Finite Volume Scheme



$$\partial C_{ij}^n = \tilde{\mathbf{x}}(\chi,\tau) = \sum_{k=1}^{4} \beta_k(\chi,\tau)\tilde{\mathbf{X}}_{ij,k}^n$$

$$0 \le \chi \le 1, \quad 0 \le \tau \le 1$$

$$|\partial C_{ij}^n| = \left|\frac{\partial \tilde{\mathbf{x}}}{\partial \chi} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tau}\right|, \quad \tilde{\mathbf{n}}_{ij} = \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \chi} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tau}\right) / |\partial C_{ij}^n|$$

ALE–type One–Step Finite Volume Scheme

$$|T_i^{n+1}|\,\mathbf{Q}_i^{n+1} = |T_i^n|\,\mathbf{Q}_i^n - \sum_{T_j \in \mathcal{N}_i} \int_0^1 \int_0^1 |\partial C_{ij}^n| \tilde{\mathbf{Q}}_{ij} \cdot \tilde{\mathbf{n}}_{ij}\, d\chi d\tau + \int_{t^n}^{t^{n+1}} \int_{T_i(t)} \mathbf{S}(\mathbf{q}_h)\, d\mathbf{x} dt$$

Finite Volume Scheme

$$|T_i^{n+1}|\,\mathbf{Q}_i^{n+1} = |T_i^n|\,\mathbf{Q}_i^n - \sum_{T_j \in \mathcal{N}_i} \int_0^1 \int_0^1 |\partial C_{ij}^n| \tilde{\mathbf{Q}}_{ij} \cdot \tilde{\mathbf{n}}_{ij}\, d\chi d\tau + \int_{t^n}^{t^{n+1}} \int_{T_i(t)} \mathbf{S}(\mathbf{q}_h)\, d\mathbf{x} dt$$

Rusanov numerical flux

$$\tilde{\mathbf{Q}}_{ij} \cdot \tilde{\mathbf{n}}_{ij} = \frac{1}{2}\left(\tilde{\mathbf{Q}}(\mathbf{q}_h^+) + \tilde{\mathbf{Q}}(\mathbf{q}_h^-)\right) \cdot \tilde{\mathbf{n}}_{ij} - \frac{1}{2} s_{\max}\left(\mathbf{q}_h^+ - \mathbf{q}_h^-\right)$$

$$\mathbf{A}_{\mathbf{n}}^{\mathbf{V}}(\mathbf{Q}) = \partial(\mathbf{F} \cdot \mathbf{n})/\partial\mathbf{Q} - (\mathbf{V} \cdot \mathbf{n})\,\mathbf{I}$$

with $\quad \mathbf{n} = \dfrac{(\tilde{n}_x, \tilde{n}_y)^T}{\sqrt{\tilde{n}_x^2 + \tilde{n}_y^2}} \quad$ and $\quad \mathbf{V} \cdot \mathbf{n} = -\dfrac{\tilde{n}_t}{\sqrt{\tilde{n}_x^2 + \tilde{n}_y^2}}$

Finite Volume Scheme

$$|T_i^{n+1}|\,\mathbf{Q}_i^{n+1} = |T_i^n|\,\mathbf{Q}_i^n - \sum_{T_j\in\mathcal{N}_i} \int_0^1 \int_0^1 |\partial C_{ij}^n| \tilde{\mathbf{Q}}_{ij}\cdot\tilde{\mathbf{n}}_{ij}\, d\chi d\tau + \int_{t^n}^{t^{n+1}} \int_{T_i(t)} \mathbf{S}(\mathbf{q}_h)\, d\mathbf{x}dt$$

Osher numerical flux

$$\tilde{\mathbf{Q}}_{ij}\cdot\tilde{\mathbf{n}}_{ij} = \frac{1}{2}\left(\tilde{\mathbf{Q}}(\mathbf{q}_h^+) + \tilde{\mathbf{Q}}(\mathbf{q}_h^-)\right)\cdot\tilde{\mathbf{n}}_{ij} - \frac{1}{2}\left(\int_0^1 \left|\mathbf{A}_{\mathbf{n}}^{\mathbf{V}}(\boldsymbol{\Psi}(s))\right| ds\right)\left(\mathbf{q}_h^+ - \mathbf{q}_h^-\right)$$

$$\boldsymbol{\Psi}(s) = \mathbf{q}_h^- + s\left(\mathbf{q}_h^+ - \mathbf{q}_h^-\right), \qquad 0 \le s \le 1$$

The Euler equations

$$\mathbf{Q}_t + \mathbf{f}_x + \mathbf{g}_y = \mathbf{S}(x, y, t)$$

with

$$\mathbf{Q} = \begin{pmatrix} \rho \\ \rho v_x \\ \rho v_y \\ \rho E \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \rho v_x \\ \rho v_x^2 + p \\ \rho v_x v_y \\ v_x(\rho E + p) \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \rho v_y \\ \rho v_y v_x \\ \rho v_y^2 + p \\ v_y(\rho E + p) \end{pmatrix}.$$

$\rho \rightarrow$ fluid density

$\mathbf{v} = (v_x, v_y) \rightarrow$ fluid's velocity vector

$\mathbf{S} \rightarrow$ vector of source term ($\mathbf{S} = 0$)

$p \rightarrow$ fluid pressure

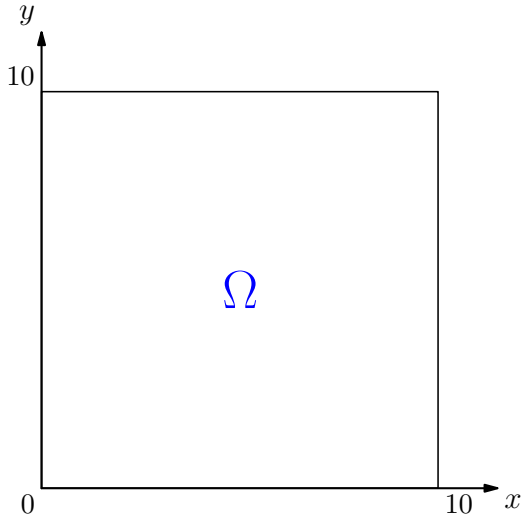
$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho(v_x^2 + v_y^2)\right)$$

$\downarrow$

equation of state

Convergence studies: *Shu* Vortex

IC: $(\rho, v_x, v_y, p) = (1+\delta\rho, 1+\delta v_x, 1+\delta v_y, 1+\delta p)$



Computational domain

$\Omega(0) = [0;10] \times [0;10]$

Convective velocity

$\mathbf{v}_c = (1,1)$

Perturbations

$$\begin{aligned} \begin{bmatrix} \delta v_x \\ \delta v_y \end{bmatrix} &= \frac{\epsilon}{2\pi} e^{\frac{1-r^2}{2}} \begin{bmatrix} -(y-5) \\ (x-5) \end{bmatrix} \\ \delta S &= 0 \\ \delta T &= -\frac{(\gamma-1)\epsilon^2}{8\gamma\pi^2} e^{1-r^2} \end{aligned}$$

Parameters

$$\begin{aligned} r^2 &= (x-5)^2 + (y-5)^2 \text{ radius} \\ \epsilon &= 5 \text{ vortex strength} \\ \gamma &= 1.4 \text{ ratio of specific heat} \end{aligned}$$

Convergence studies: *Shu* Vortex

$h(\Omega(t_f))$	$\mathcal{NS}_{cs}$ ϵ_{L_2}	$\mathcal{O}(L_2)$	$\mathcal{NS}_m$ ϵ_{L_2}	$\mathcal{O}(L_2)$	$\mathcal{NS}_b$ ϵ_{L_2}	$\mathcal{O}(L_2)$
			$\mathcal{O}1$			
3.73E-01	6.9161E-02	-	6.9535E-02	-	7.0045E-02	-
2.62E-01	5.0030E-02	0.92	5.0715E-02	0.82	5.1427E-02	0.99
1.76E-01	3.4607E-02	0.92	3.5310E-03	0.90	3.5513E-02	0.95
1.36E-01	2.5724E-02	1.14	2.6151E-03	1.11	2.6277E-02	1.07
			$\mathcal{O}2$			
3.49E-01	4.0451E-02	-	3.9160E-02	-	3.9585E-02	-
2.49E-01	2.6261E-02	1.28	2.5787E-02	1.34	2.5758E-02	1.44
1.69E-01	1.5884E-02	1.29	1.5721E-03	1.24	1.5625E-02	1.24
1.28E-01	1.0355E-02	1.55	1.0300E-03	1.58	1.0258E-02	1.54
			$\mathcal{O}3$			
3.28E-01	1.6140E-02	-	1.6172E-02	-	1.6206E-02	-
2.51E-01	6.9455E-03	3.16	6.9570E-03	3.10	6.9638E-03	3.15
1.68E-01	2.2904E-03	2.75	2.2913E-03	2.74	2.2925E-03	2.75
1.28E-01	9.2805E-04	3.33	9.2763E-04	3.34	9.2806E-04	3.33
			$\mathcal{O}4$			
3.29E-01	4.4613E-03	-	4.4636E-03	-	4.4627E-03	-
2.51E-01	1.7186E-03	3.54	1.7189E-03	3.53	1.7188E-03	3.55
1.68E-01	4.2840E-04	3.43	4.2834E-04	3.44	4.2845E-03	3.43
1.28E-01	1.3480E-04	4.27	1.3477E-04	4.27	1.3483E-04	4.27
			$\mathcal{O}5$			
3.29E-01	4.4850E-03	-	4.4811E-03	-	4.4823E-03	-
2.51E-01	1.2711E-03	4.65	1.2705E-03	4.63	1.2705E-03	4.66
1.68E-01	2.2531E-04	4.28	2.2516E-04	4.28	2.2520E-03	4.27
1.28E-01	5.7869E-05	5.02	5.7821E-05	5.02	5.7838E-04	5.02

The Kidder Problem

IC:

$$
\begin{aligned}
\rho_0 &= \rho(r,0) = \left(\frac{r_{e,0}^2 - r^2}{r_{e,0}^2 - r_{i,0}^2} \rho_{i,0}^{\gamma-1} + \frac{r^2 - r_{i,0}^2}{r_{e,0}^2 - r_{e,0}^2} \rho_{e,0}^{\gamma-1} \right)^{\frac{1}{\gamma-1}} \\
s_0 &= \frac{p_0}{\rho_0^\gamma} = 1 \\
p_0(r) &= s_0 \rho_0(r)^\gamma
\end{aligned}
$$

Parameters

$$
\begin{aligned}
r_e(0) &= 1.0 \text{ external radius} \\
r_i(0) &= 0.9 \text{ internal radius} \\
\gamma &= 2.0 \text{ ratio of specific heat} \\
\rho_{i,0} &= 1.0 \text{ initial internal density} \\
\rho_{e,0} &= 2.0 \text{ initial external density}
\end{aligned}
$$

The Kidder Problem

Analytical Solution [6]

$$
\begin{aligned}
\rho\left(R(r,t),t\right) &= h(t)^{-\frac{2}{\gamma-1}}\rho_0\left[\frac{R(r,t)}{h(t)}\right] \\
u_r\left(R(r,t),t\right) &= \frac{d}{dt}h(t)\left[\frac{R(r,t)}{h(t)}\right] \\
p\left(R(r,t),t\right) &= h(t)-\frac{2\gamma}{\gamma-1}p_0\left[\frac{R(r,t)}{h(t)}\right]
\end{aligned}
$$

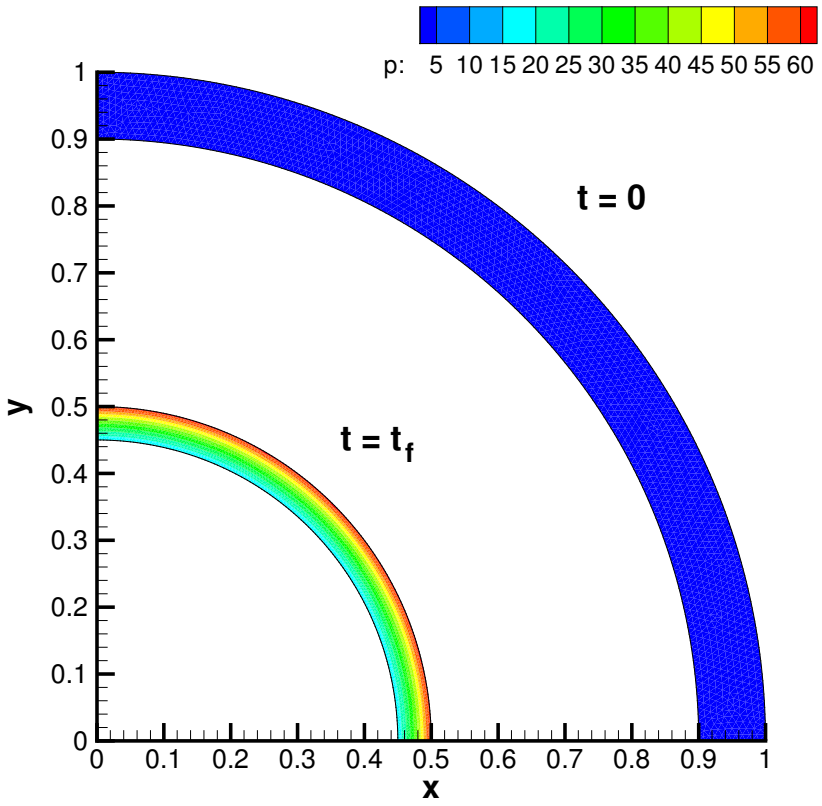
with

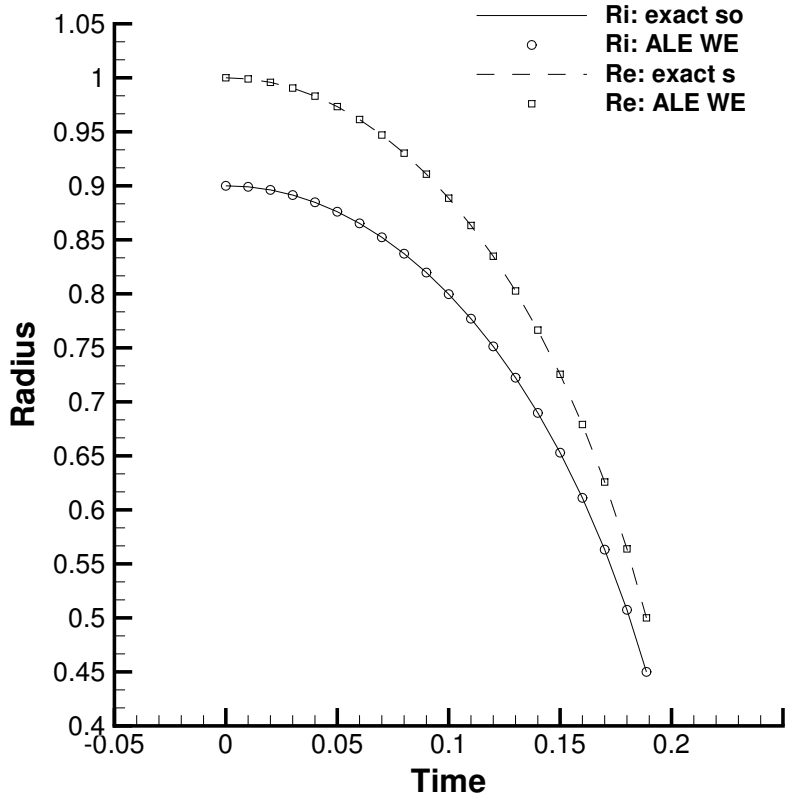
$$
h(t)=\sqrt{1-\frac{t^2}{\tau^2}}, \quad \tau=\sqrt{\frac{\gamma-1}{2}\frac{(r_{e,0}^2-r_{i,0}^2)}{c_{e,0}^2-c_{i,0}^2}}
$$

and the internal and external sound speeds c_i and c_e defined as

$$
c_i=\sqrt{\gamma\frac{p_i}{\rho_i}}, \qquad c_e=\sqrt{\gamma\frac{p_e}{\rho_e}}
$$

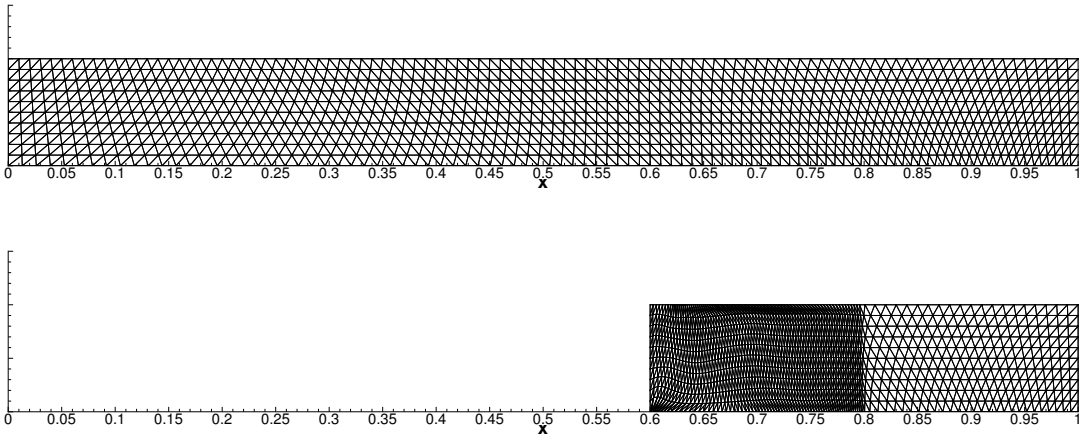
The Kidder Problem





	$\mathcal{NS}_{cs}$	$\mathcal{NS}_m$	$\mathcal{NS}_b$
e_{int}	7.72443E-06	7.72460E-06	7.73181E-06
e_{ext}	1.01812E-05	1.01811E-05	1.01867E-05

The Saltzman Problem



Computational domain
$\Omega(0) = [0;1] \times [0;0.1]$

Piston velocity
$\mathbf{v}_p = (1,0)$

Mesh size
100×10

IC:
$\rho_0 = 1,\ \mathbf{v}_0 = \mathbf{0},\ e_0 = 10^{-4}$

Analytical solution

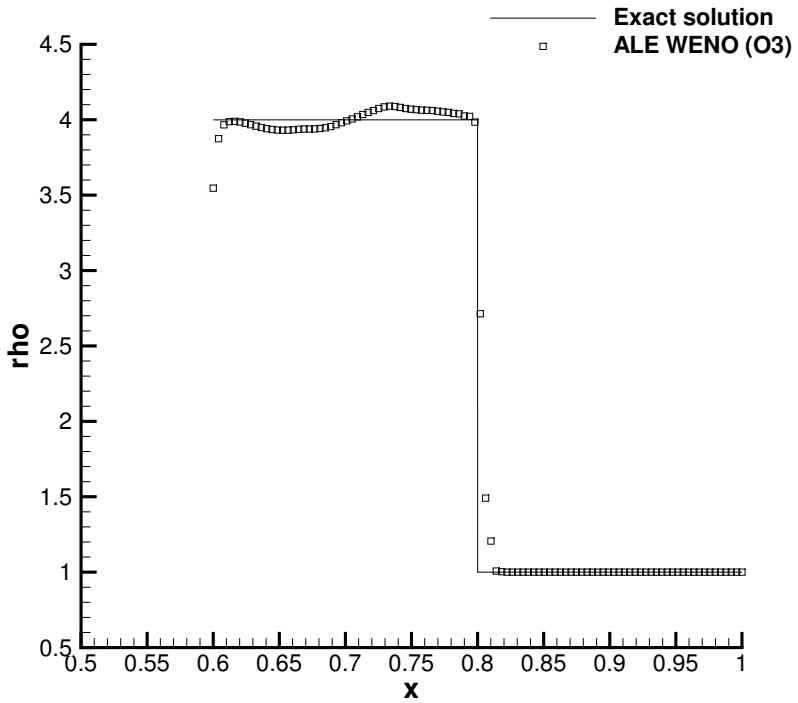
	Left state	*Right state*
ρ	1.0	1.0
u	1.0	-1.0
v	0.0	0.0
p	$6.67 \cdot 10^{-5}$	$6.67 \cdot 10^{-5}$

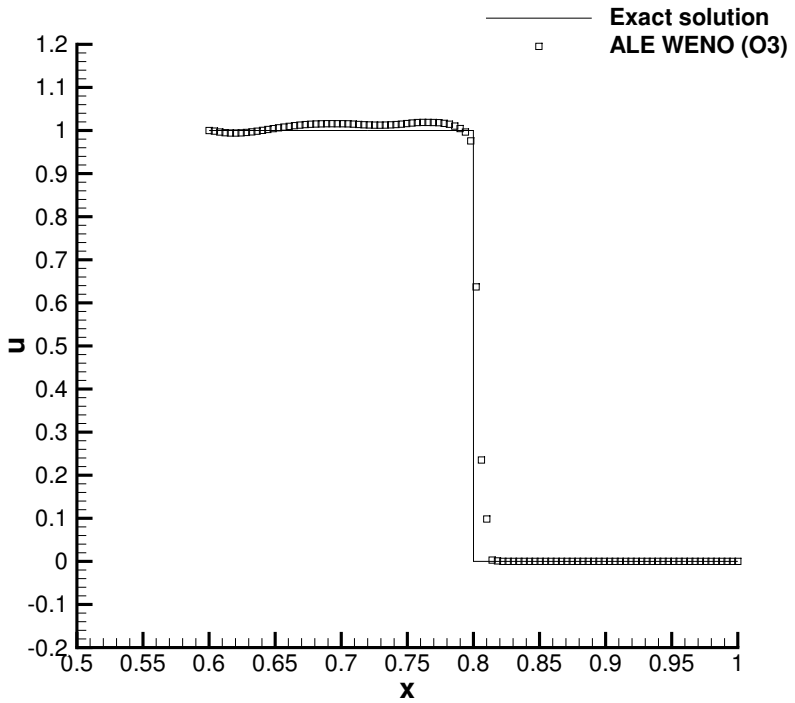
final time:	$t_f = 0.6$
shifting:	$d = u_p \cdot t_f$
final shock location:	$x = 0.8$
exact density:	$\rho_e = 4.0$

The Saltzman Problem

Saltzman Test

The Saltzman Problem





3rd order numerical results of density and velocity for the Saltzman problem.

The ideal MHD equations[2]

$$\mathbf{Q}_t + \mathbf{f}_x + \mathbf{g}_y = \mathbf{S}(x, y, t)$$

with

$$\mathbf{Q} = (\rho, \rho v_x, \rho v_y, \rho E, B_x, B_y, \Psi)^T, \qquad \mathbf{S} = 0,$$

$$\mathbf{f} = \begin{pmatrix} \rho v_x \\ \rho v_x v_x + p_{tot} - \frac{B_x B_x}{4\pi} \\ \rho v_y v_x - \frac{B_y B_x}{4\pi} \\ (\rho E + p_{tot})\, v_x - \frac{B_x(\mathbf{v}\cdot\mathbf{B})}{4\pi} \\ B_x v_x - v_x B_x + \Psi \\ B_y v_x - v_y B_x \\ c_h^2 B_x \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} \rho v_y \\ \rho v_x v_y - \frac{B_x B_y}{4\pi} \\ \rho v_y v_y + p_{tot} - \frac{B_y B_y}{4\pi} \\ (\rho E + p_{tot})\, v_y - \frac{B_y(\mathbf{v}\cdot\mathbf{B})}{4\pi} \\ B_x v_y - v_x B_y \\ B_y v_y - v_y B_y + \Psi \\ c_h^2 B_y \end{pmatrix},$$

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho(v_x^2 + v_y^2) - \frac{(B_x^2 + B_y^2)}{8\pi}\right).$$

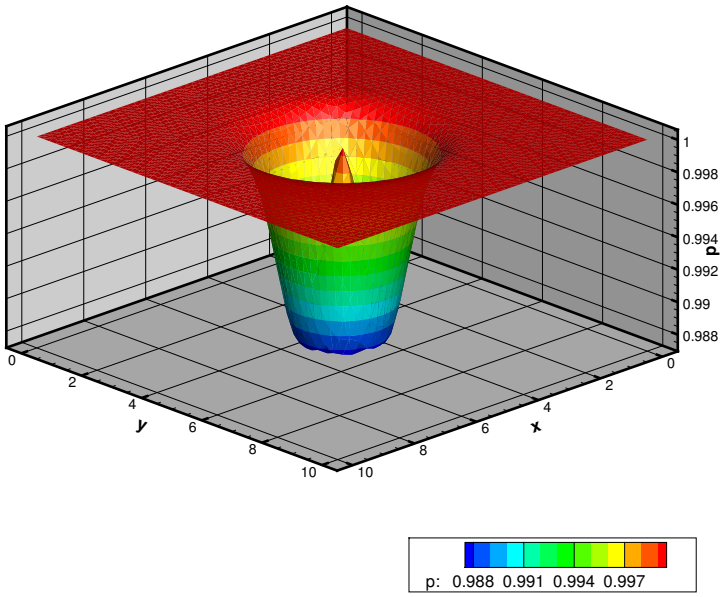
[2] divergence cleaning approach is used [Dedner et al., *JCP* 2002].

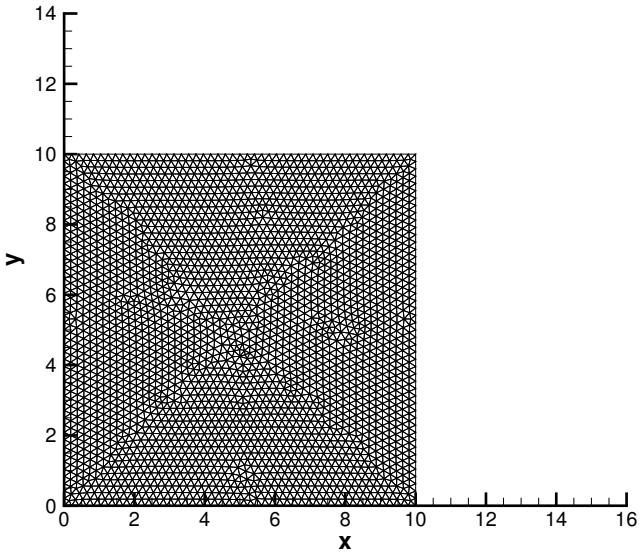
Convergence studies: MHD Vortex

IC: $(\rho, v_x, v_y, p, B_x, B_y, \Psi) = (1, 1+\delta v_x, 1+\delta v_y, 1+\delta p, \delta B_x, \delta B_y, 0)$

$$\begin{bmatrix} \delta v_x \\ \delta v_y \\ \delta p \\ \delta B_x \\ \delta B_y \end{bmatrix} = \begin{bmatrix} \frac{\epsilon}{2\pi} e^{\frac{1}{2}(1-r^2)}(5-y) \\ \frac{\epsilon}{2\pi} e^{\frac{1}{2}(1-r^2)}(x-5) \\ \frac{1}{8\pi}\left(\frac{\mu}{2\pi}\right)^2 (1-r^2) e^{(1-r^2)} - \frac{1}{2}\left(\frac{\epsilon}{2\pi}\right)^2 e^{(1-r^2)} \\ \frac{\mu}{2\pi} e^{\frac{1}{2}(1-r^2)}(5-y) \\ \frac{\mu}{2\pi} e^{\frac{1}{2}(1-r^2)}(x-5) \end{bmatrix}$$

Convergence studies: MHD Vortex





Computational domain
$\Omega(0) = [0; 10] \times [0; 10]$

Convective velocity
$\mathbf{v}_c = (1, 1)$

Parameters

$$
\begin{aligned}
r^2 &= (x-5)^2 + (y-5)^2 \text{ radius} \\
\epsilon &= 1 \text{ vortex strength} \\
\mu &= \sqrt{4\pi} \\
\gamma &= \frac{5}{3} \text{ ratio of specific heat}
\end{aligned}
$$

Convergence studies: MHD Vortex

$h(\Omega(t_f))$	$\mathcal{NS}_{cs}$ ϵ_{L_2}	$\mathcal{O}(L_2)$	$\mathcal{NS}_m$ ϵ_{L_2}	$\mathcal{O}(L_2)$	$\mathcal{NS}_b$ ϵ_{L_2}	$\mathcal{O}(L_2)$
			$\mathcal{O}1$			
3.26E-01	2.7330E-03	-	2.7059E-03	-	2.7381E-03	-
2.37E-01	2.0111E-03	0.96	2.0173E-03	0.90	2.0173E-03	0.93
1.64E-01	1.3081E-03	1.17	1.3055E-03	1.20	1.3113E-03	1.20
1.28E-01	9.5497E-04	1.26	9.5150E-04	1.30	9.5617E-04	1.28
			$\mathcal{O}2$			
3.26E-01	4.8091E-03	-	4.7707E-03	-	5.5971E-03	-
2.35E-01	2.8382E-03	1.61	2.8571E-03	1.58	2.7874E-03	2.13
1.64E-01	1.4212E-03	1.91	1.4239E-03	1.88	1.3789E-03	1.94
1.28E-01	6.4686E-04	3.24	6.4610E-04	3.26	7.2141E-04	2.67
			$\mathcal{O}3$			
3.25E-01	1.1417E-03	-	1.1376E-03	-	1.1265E-03	-
2.36E-01	1.8935E-04	5.57	1.8930E-04	5.56	1.8632E-04	5.56
1.63E-01	7.1734E-05	2.65	7.1740E-05	2.65	7.1912E-05	2.60
1.28E-01	3.1651E-05	3.38	3.1653E-05	3.38	3.1738E-05	3.38
			$\mathcal{O}4$			
3.26E-01	2.4858E-04	-	2.4864E-04	-	2.4472E-04	-
2.35E-01	7.9871E-05	3.50	7.9875E-05	3.50	7.9884E-05	3.45
1.63E-01	2.1790E-05	3.55	2.1791E-05	3.55	2.1795E-05	3.55
1.28E-01	8.2013E-06	4.03	8.2014E-06	4.03	8.1998E-06	4.03
			$\mathcal{O}5$			
3.26E-01	1.2010E-04	-	1.2010E-04	-	1.1992E-04	-
2.35E-01	2.7365E-05	4.56	2.7359E-05	4.56	2.7327E-05	4.56
1.63E-01	4.8779E-06	4.71	4.8778E-06	4.71	4.8898E-06	4.70
1.28E-01	1.3947E-06	5.17	1.3947E-06	5.17	1.3935E-06	5.18

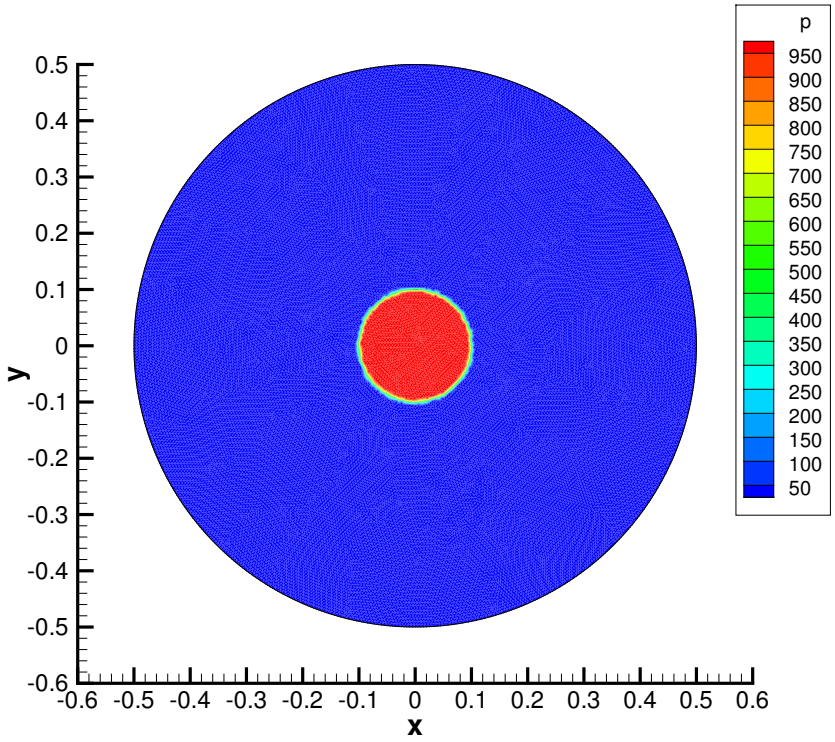
The ideal MHD Blast Problem

IC: $(\rho, v_x, v_y, p, B_x, B_y, \Psi) = (1, 0, 0, p^*, 70, 0, 0)$

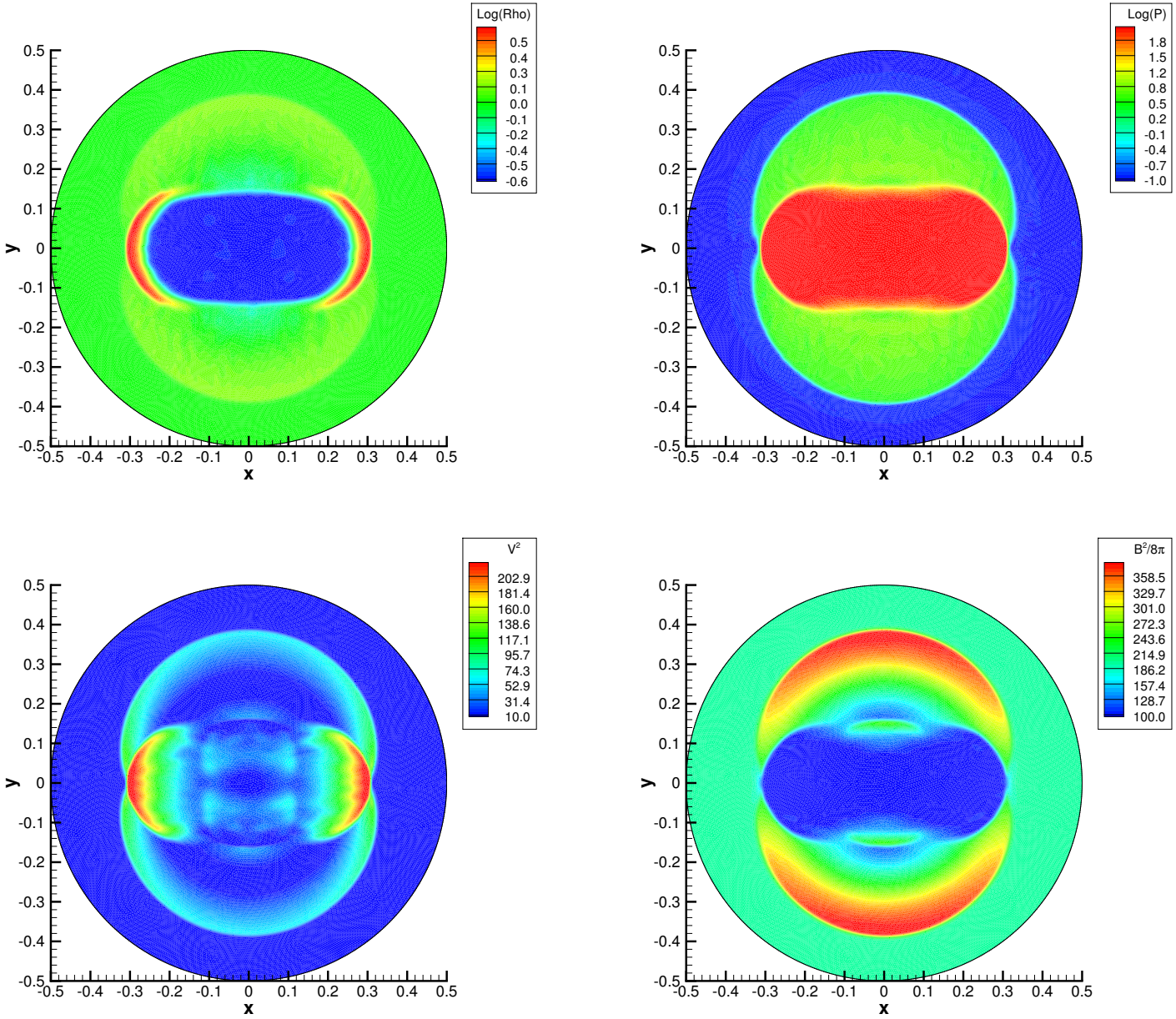
$$p^* = \begin{cases} 0 & r > r_0 \\ 1000 & r \leq r_0 \end{cases} \qquad r_0 = 0.1$$

Computational domain
$\Omega(0) = \Omega(r, \phi) = \left[\frac{1}{2}, 2\pi\right]$

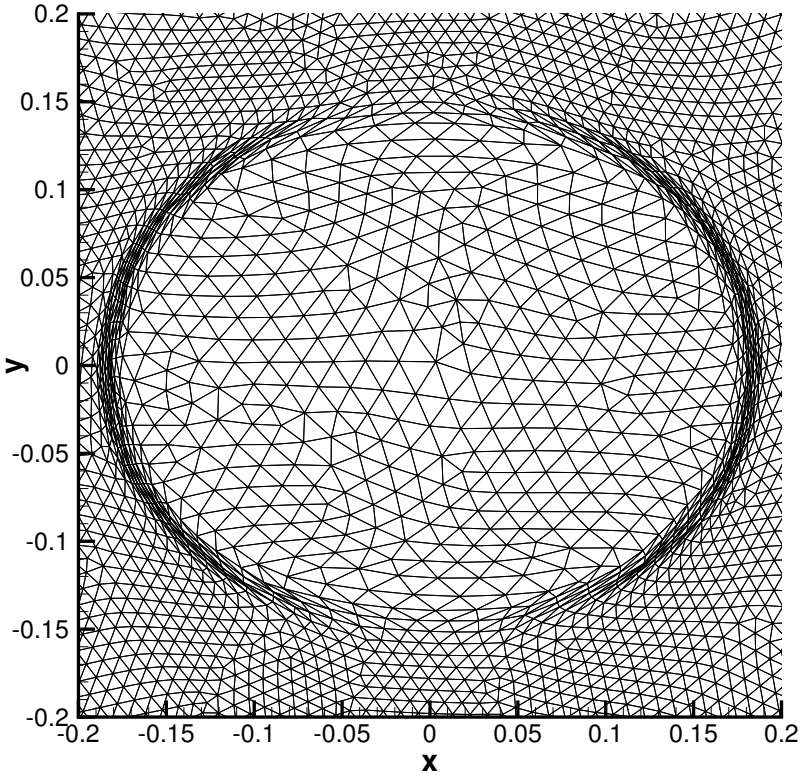
$t_f = 0.001$

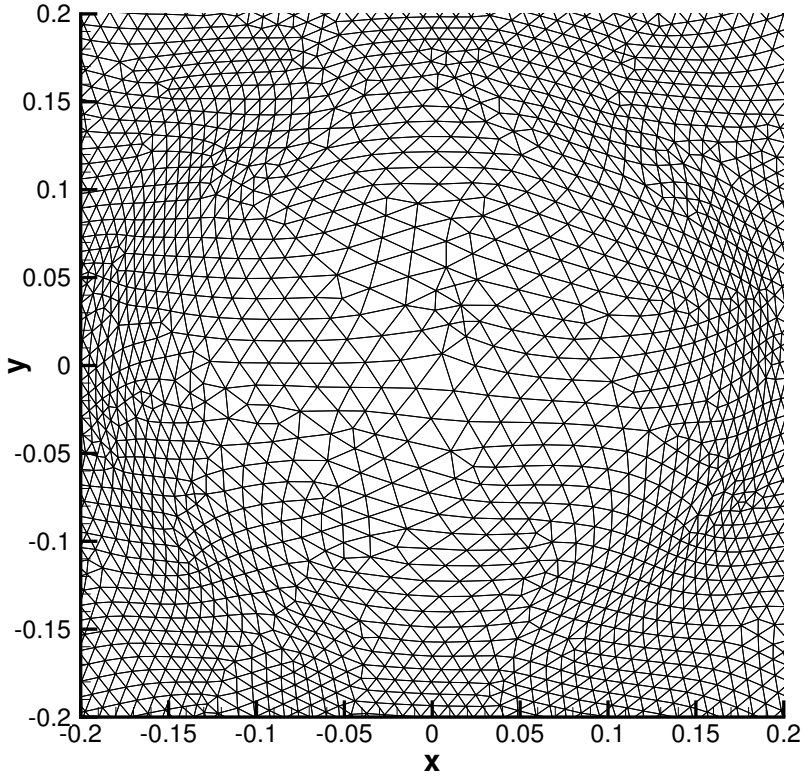


The ideal MHD Blast Problem



The ideal MHD Blast Problem





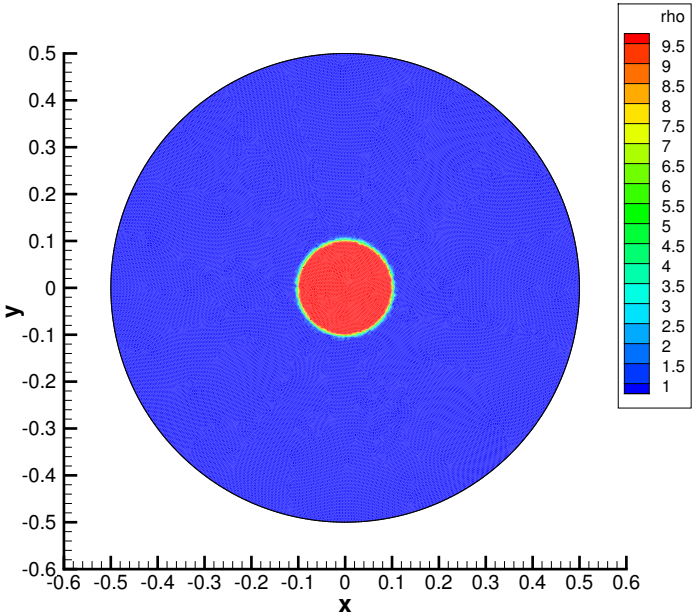
Mesh configurations for the Blast problem at time $t = 0.0004$ without (left) and with (right) rezoning.

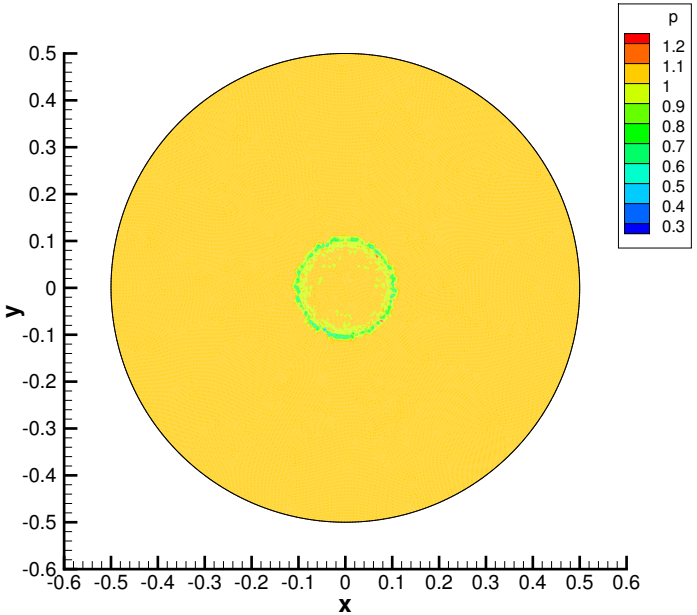
The ideal MHD Rotor Problem

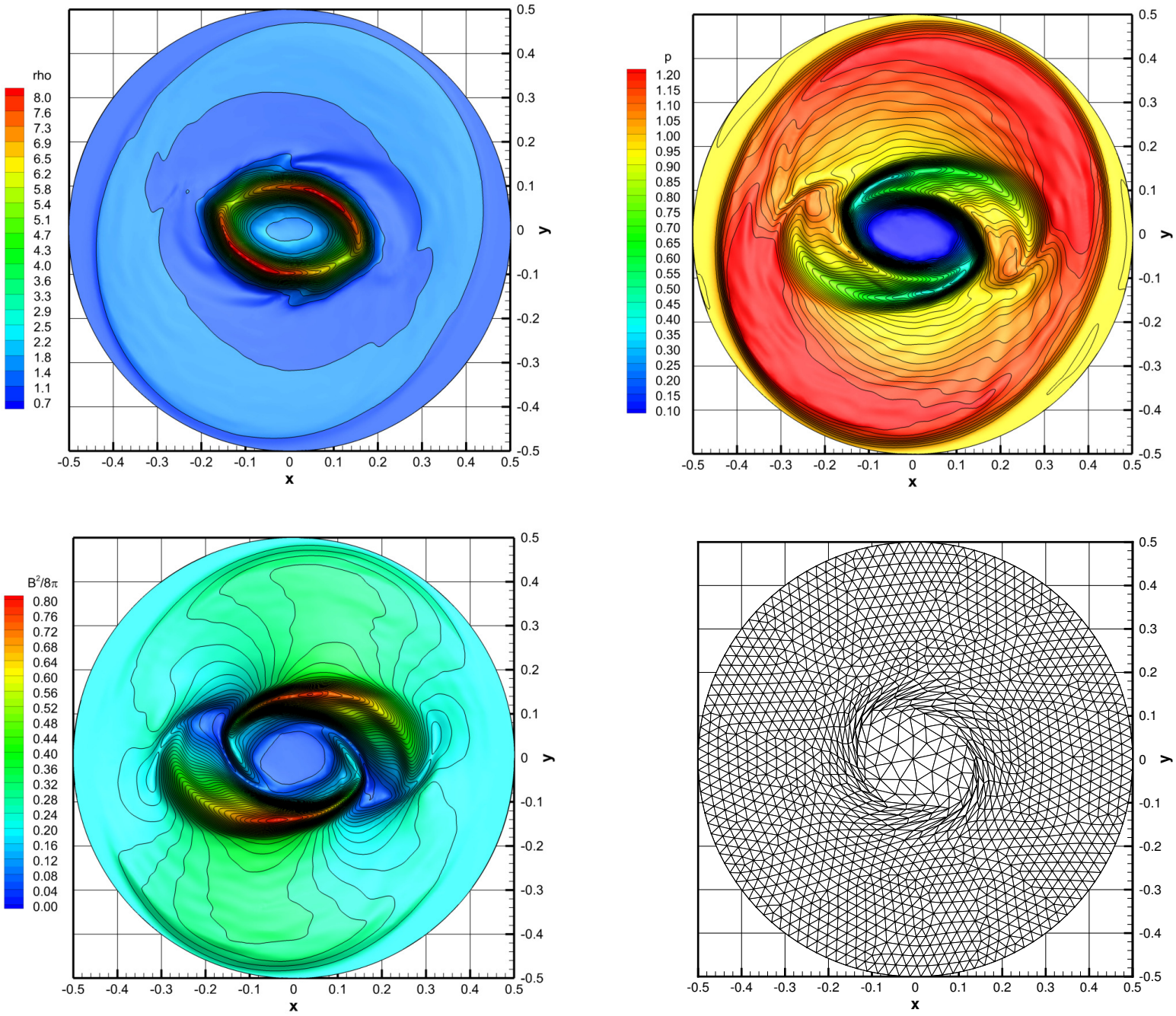
IC: $[\rho, v_x, v_y, p, B_x, B_y, \Psi]$ $\qquad \Omega(0) = \Omega(r,\phi) = \left[\frac{1}{2}, 2\pi\right]$ $\qquad t_f = 0.25$

$$\text{IC} = \begin{cases} [1,0,0,1,2.5,0,0] & r > r_1 \\ \left[\tau, -\tau\left(\frac{y}{r}\right), \tau\left(\frac{x}{r}\right), 2.5\cdot\tau, 0, 0\right] & r_0 < r \le r_1 \\ \left[10, -\left(\frac{r}{r_0}\right)\left(\frac{y}{r}\right), \left(\frac{r}{r_0}\right)\left(\frac{x}{r}\right), 1, 2.5, 0, 0\right] & r \le r_0 \end{cases}$$

$$\tau = \frac{1-(r-r_0)}{r_1 - r_0}, \qquad r_0 = 0.1, \qquad r_1 = 0.125.$$






rho
8.0
7.6
7.3
6.9
6.5
6.2
5.8
5.4
5.1
4.7
4.3
4.0
3.6
3.3
2.9
2.5
2.2
1.8
1.4
1.1
0.7
p
1.20
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
0.50
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
B²/8π
0.80
0.76
0.72
0.68
0.64
0.60
0.56
0.52
0.48
0.44
0.40
0.36
0.32
0.28
0.24
0.20
0.16
0.12
0.08
0.04
0.00
x
y

Conclusions and Outlook

Conclusions

- we showed high order ALE WENO FV schemes;
- three different node solvers have been used;
- for the first time MHD equations have been treated with the ALE WENO FV method.

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- we showed high order ALE WENO FV schemes;
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- for the first time MHD equations have been treated with the ALE WENO FV method.

Outlook

- extension of the algorithm to the 3D case;
- usage of the multidimensional node solver $\mathcal{NS}_b$ also in the flux evaluation.

Thank you!

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[1] D.S. Balsara, M. Dumbser, and R. Abgrall.
Multidimensional hllc riemann solver for unstructured meshes.
Journal of Computational Physics.
submitted to.

[2] W. Boscheri and M. Dumbser.
Arbitrary-lagrangian-eulerian one-step weno finite volume schemes on unstructured triangular meshes.
Communications in Computational Physics.
in press.

[3] J. Cheng and C.W. Shu.
A high order ENO conservative Lagrangian type scheme for the compressible Euler equations.
Journal of Computational Physics, 227:1567–1596, 2007.

[4] M. Dumbser and W. Boscheri.
High–order unstructured lagrangian one–step weno finite volume schemes for non–conservative hyperbolic systems: Applications to compressible multi–phase flows.
Computers and Fluids.
in press.

[5] M. Dumbser and M. Käser.
Arbitrary high order non-oscillatory finite volume schemes on unstructured meshes for linear hyperbolic systems.
Journal of Computational Physics, 221:693–723, 2007.

[6] R.E. Kidder.
Laser-driven compression of hollow shells: power requirements and stability limitations.
Nucl. Fus., 1:3 – 14, 1976.

[7] P.H. Maire.
A high-order cell-centered lagrangian scheme for two-dimensional compressible fluid flows on unstructured meshes.
Journal of Computational Physics, 228:2391 – 2425, 2009.

The Baer-Nunziato model

$$
\left.
\begin{aligned}
&\frac{\partial}{\partial t}\left(\phi_1\rho_1\right) + \nabla\cdot\left(\phi_1\rho_1\mathbf{u_1}\right) = 0 \\
&\frac{\partial}{\partial t}\left(\phi_1\rho_1\mathbf{u_1}\right) + \nabla\cdot\left(\phi_1\rho_1\mathbf{u_1}\mathbf{u_1}\right) + \nabla\phi_1 p_1 = p_I\nabla\phi_1 - \lambda\left(\mathbf{u_1}-\mathbf{u_2}\right) \\
&\frac{\partial}{\partial t}\left(\phi_1\rho_1 E_1\right) + \nabla\cdot\left(\left(\phi_1\rho_1 E_1 + \phi_1 p_1\right)\mathbf{u_1}\right) = -p_I\partial_t\phi_1 - \lambda\,\mathbf{u_I}\cdot\left(\mathbf{u_1}-\mathbf{u_2}\right) \\
&\frac{\partial}{\partial t}\left(\phi_2\rho_2\right) + \nabla\cdot\left(\phi_2\rho_2\mathbf{u_2}\right) = 0 \\
&\frac{\partial}{\partial t}\left(\phi_2\rho_2\mathbf{u_2}\right) + \nabla\cdot\left(\phi_2\rho_2\mathbf{u_2}\mathbf{u_2}\right) + \nabla\phi_2 p_2 = p_I\nabla\phi_2 - \lambda\left(\mathbf{u_2}-\mathbf{u_1}\right) \\
&\frac{\partial}{\partial t}\left(\phi_2\rho_2 E_2\right) + \nabla\cdot\left(\left(\phi_2\rho_2 E_2 + \phi_2 p_2\right)\mathbf{u_2}\right) = p_I\partial_t\phi_1 - \lambda\,\mathbf{u_I}\cdot\left(\mathbf{u_2}-\mathbf{u_1}\right) \\
&\frac{\partial}{\partial t}\phi_1 + \mathbf{u_I}\nabla\phi_1 = \nu(p_1 - p_2)
\end{aligned}
\right\}
$$

The system is closed by the stiffened gas EOS for each phase

$$
e_k = \frac{p_k + \gamma_k\pi_k}{\rho_k(\gamma_k - 1)}
$$

The Baer-Nunziato model

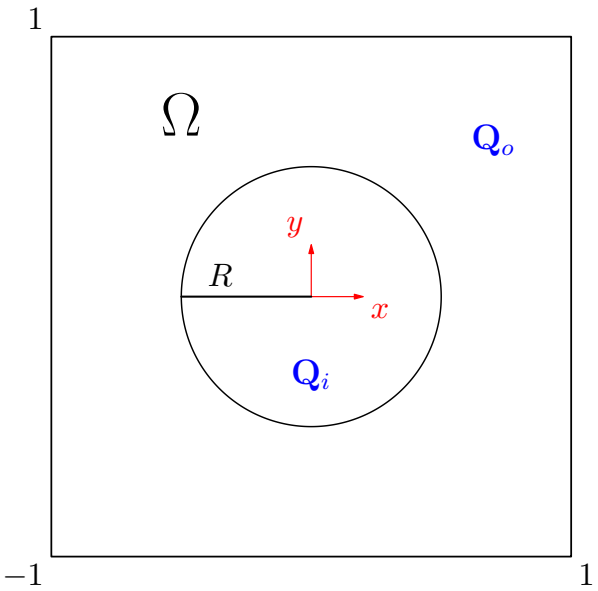
Definitions

k	$\rightarrow$	phase
ϕ_k	$\rightarrow$	volume fraction of phase k
$\mathbf{u_k}$	$\rightarrow$	velocity vector of phase k
subscripts 1	$\rightarrow$	solid phase
subscripts 2	$\rightarrow$	gas phase
subscripts I	$\rightarrow$	interface
e_k	$\rightarrow$	internal energy
$E_k = e_k + \frac{1}{2}\mathbf{u_k}^2$	$\rightarrow$	specific total energy of phase k

Assumptions

$\phi_1 + \phi_2 = 1$	volume fractions must sum to the unity
$\mathbf{u_I} = \mathbf{u_1}$	interface velocity is the solid phase velocity
$p_I = p_2$	interface pressure is the gas phase pressure

Cylindrical Explosion Problems

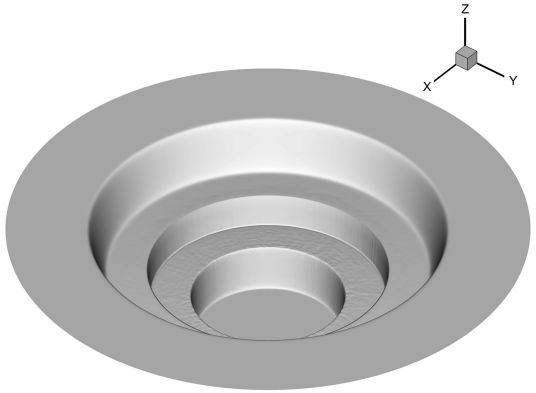


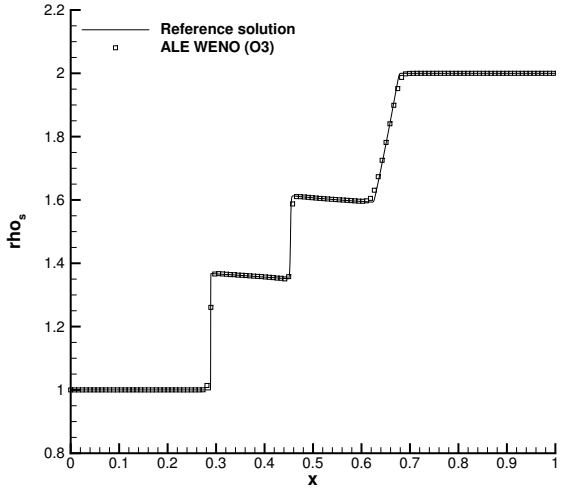
$$\mathbf{Q}(\mathbf{x},0) = \begin{cases} \mathbf{Q}_i, & \text{if} \quad |\mathbf{x}| < 0.5, \\ \mathbf{Q}_o, & \text{else.} \end{cases}$$

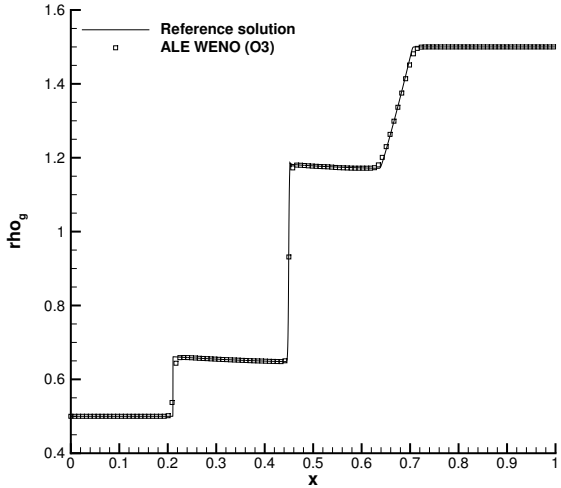
	ρ_s	u_s	p_s	ρ_g	u_g	p_g	ϕ_s	t_e
EP1	$\gamma_s = 1.4$, $\pi_s = 0$, $\gamma_g = 1.4$, $\pi_g = 0$							
$\mathbf{Q}_i$	1.0	0.0	1.0	0.5	0.0	1.0	0.4	0.10
$\mathbf{Q}_o$	2.0	0.0	2.0	1.5	0.0	2.0	0.8	
EP2	$\gamma_s = 3.0$, $\pi_s = 100$, $\gamma_g = 1.4$, $\pi_g = 0$							
$\mathbf{Q}_i$	800.0	0.0	500.0	1.5	0.0	2.0	0.4	0.10
$\mathbf{Q}_o$	1000.0	0.0	600.0	1.0	0.0	1.0	0.3	
EP3	$\gamma_s = 1.4$, $\pi_s = 0$, $\gamma_g = 1.4$, $\pi_g = 0$							
$\mathbf{Q}_i$	1.0	0.9	2.5	1.0	0.0	1.0	0.9	0.10
$\mathbf{Q}_o$	1.0	0.0	1.0	1.2	1.0	2.0	0.2	

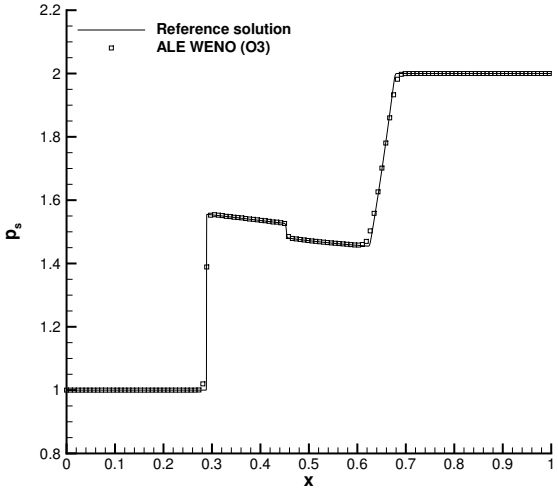
Cylindrical Explosion Problems

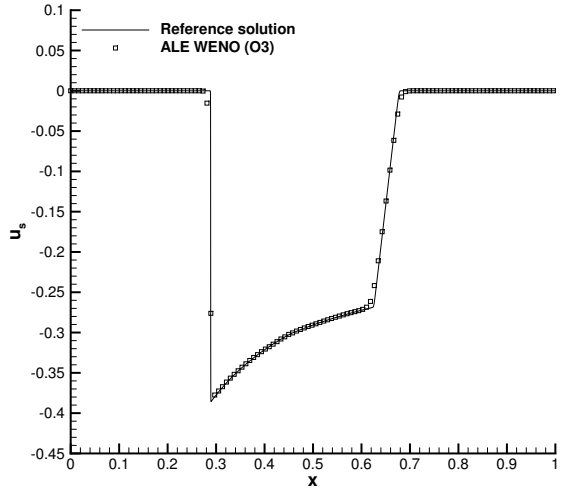
EP1 $\rightarrow t = 0.15$

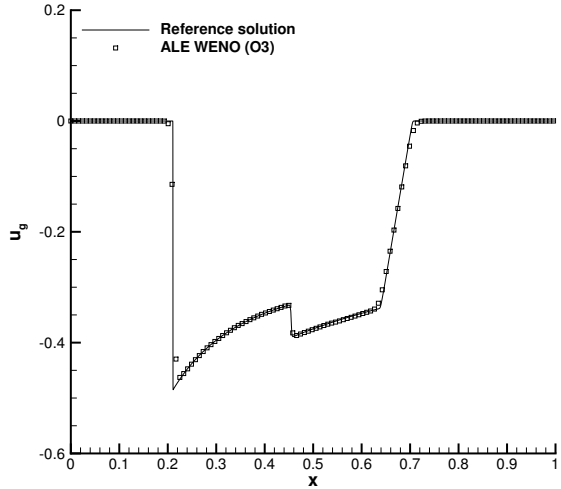






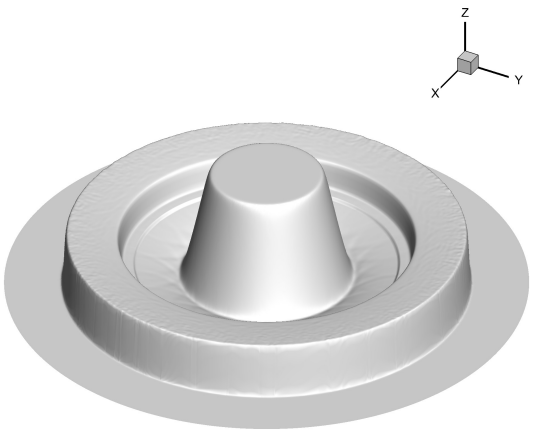


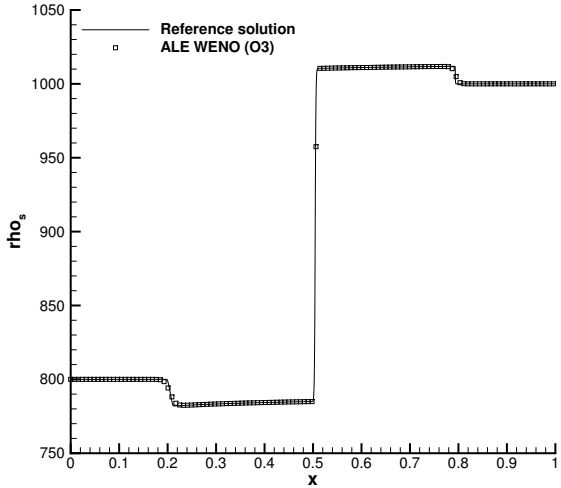


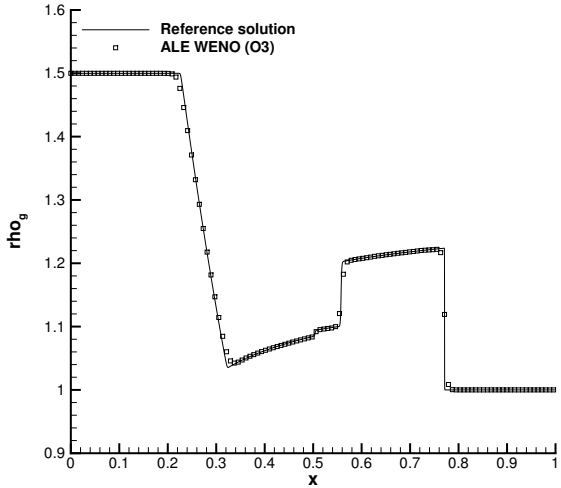


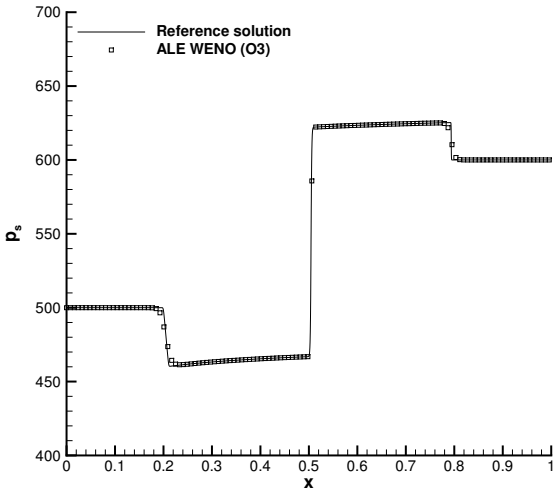
Cylindrical Explosion Problems

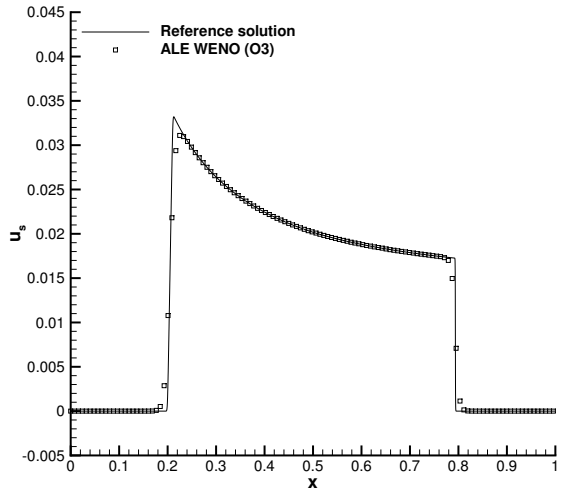
EP2 $\rightarrow t = 0.15$

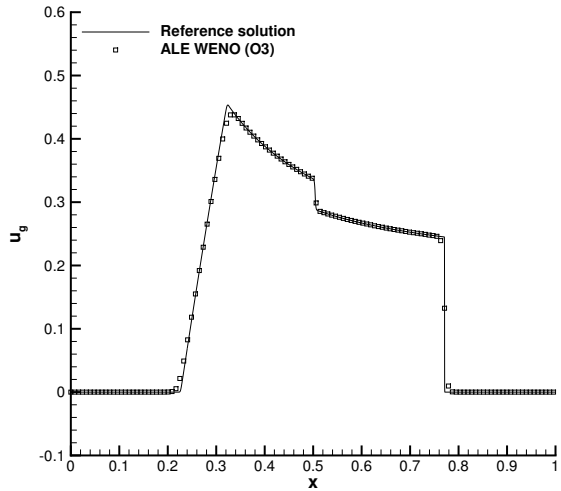


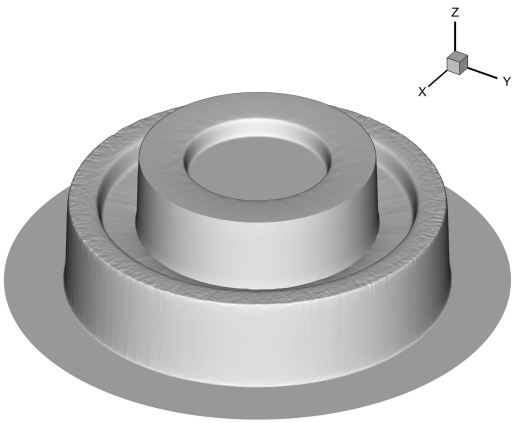
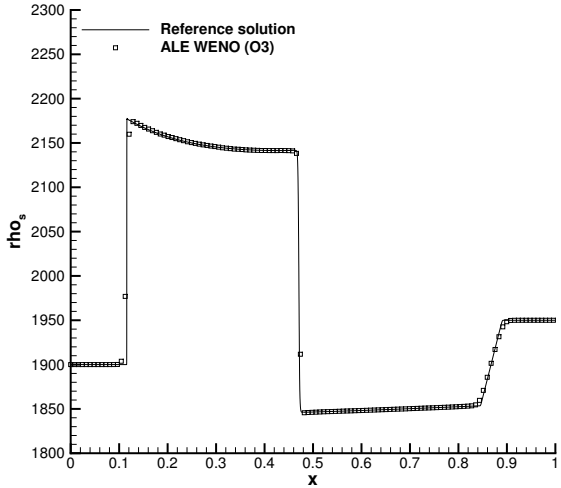
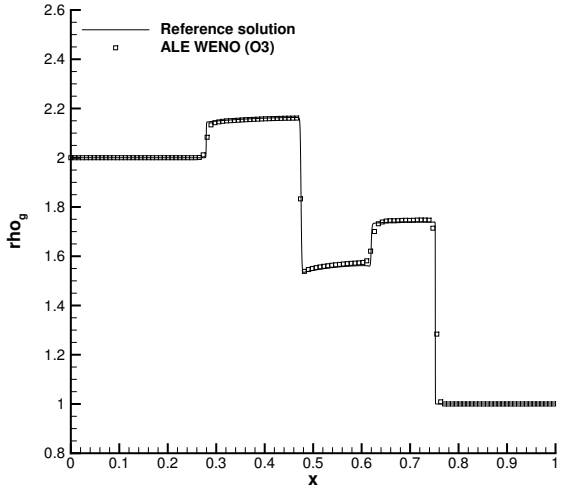
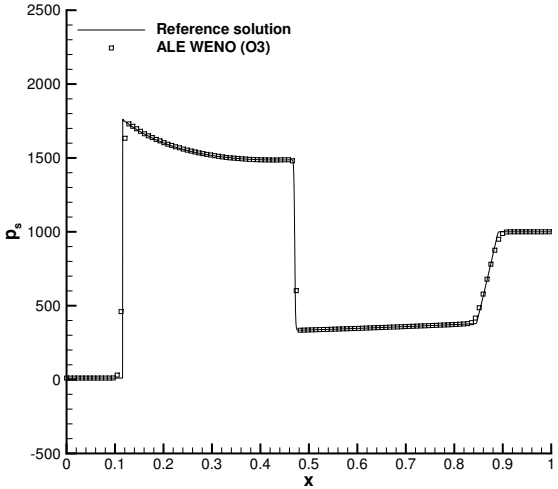
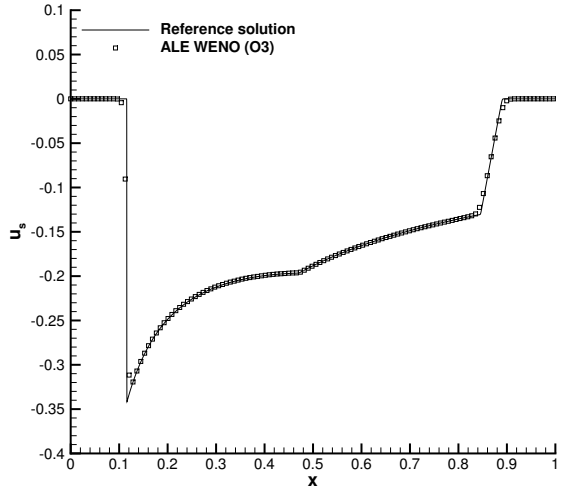
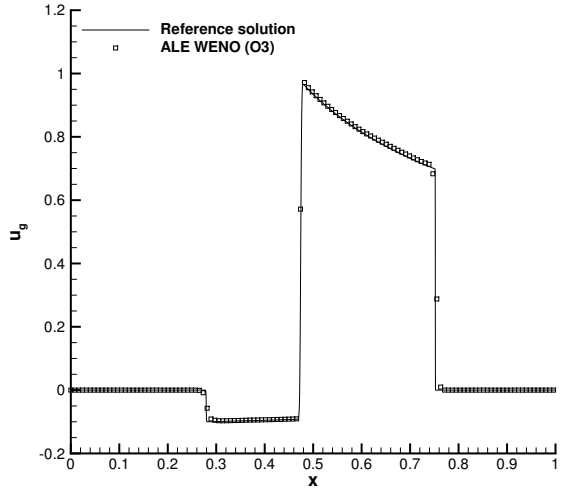










Cylindrical Explosion Problems

EP3 $\rightarrow t = 0.15$