# Low numerical dissipation Eulerian cut-cell method for coupled compressible solid/turbulent-fluid problems

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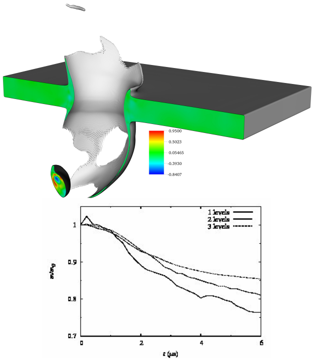
# Introduction

Motivation:

- Interested in impact and penetration problems
- Problems involving compressible solids and fluids, where fluid may become turbulent
- Simplest example (computationally) is a point source explosion in fluid above a solid surface
- Want to use modern high-order Godunov methods for fluids (and solids)

Examples of existing Eulerian sharp interface multi-material methods that provide suitable frameworks:

- Cut cell method [1] (complex,conservative)
- Ghost cell method [2] (low complexity,non-conservative)




*From [2]*

[1 ]*Barton et al., A conservative level-set based method for compressible solid/fluid problems on fixed grids, JCP (2011)*

[2 ]*Barton et al., Eulerian adaptive finite-difference method for high-velocity impact and penetration problems, JCP (2013)*

# Overview

1. Models
2. Cut-cell solver
3. Interface Tracking
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# Fluids

- It is intended to use existing methods for compressible fluid dynamics that can be categorised as **ILES**
- Therefore we solve inviscid Euler equations:

$$\begin{pmatrix} \rho \\ \rho\mathbf{u} \\ \rho E \\ \rho\psi_i \end{pmatrix}_t + \begin{pmatrix} \rho u_k \\ \rho\mathbf{u}u_k + p\mathbf{e}_k \\ \rho E u_k + \mathbf{u}\cdot\mathbf{e}_k p \\ \rho\psi_i u_k \end{pmatrix}_{x_k} = \mathbf{0}$$

*The favoured numerical methods:*

- Fixed Cartesian meshes
- Cell-centered variables
- Un-split finite difference discretisation
- Large numerical stencils
- Explicit Runge-Kutta time integration

- **However,** interested in applying low-numerical dissipation methods developed for explicit LES to solids and these can easily be switched on for fluids also
- What follows therefore forms a basis for use of LES models should they be required

# Solids: hyperelastic model

Usual mass, momentum and energy balance laws supplemented by balance laws for deformation:

$$\frac{\partial \overline{F}_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \overline{F}_{ij} - u_i \overline{F}_{kj}\right) = -u_i \beta_j, \qquad \beta_j = \frac{\partial \overline{F}_{kj}}{\partial x_k}, \qquad \overline{\mathbf{F}} = \rho \mathbf{F}.$$

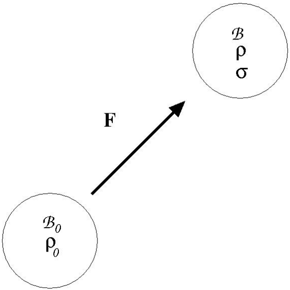
$\mathbf{F} := \partial \mathbf{x} / \partial \mathbf{x}_0$ deformation gradient
Closure relations:

- Specific internal energy

$$\mathscr{E} = \mathscr{E}(\mathbf{F}, \mathscr{S}, \mathbf{h})$$

- Cauchy stress

$$\boldsymbol{\sigma} = \rho \mathbf{F} \frac{\partial \mathscr{E}(\mathbf{F}, \mathscr{S}, \mathbf{h})}{\partial \mathbf{F}^{\mathrm{T}}}$$



# Solids: inelastic deformations

Multiplicative decomposition

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^i$$

Additional rate equations

$$D_t \mathbf{F}^i = \mathbf{L}^i \mathbf{F}^i$$

*Associative flow rules:*

$$\mathbf{L}^i = \chi \mathbf{F}^{e-1} \frac{dev\boldsymbol{\sigma}}{||dev\boldsymbol{\sigma}||} \mathbf{F}^e \chi \geq 0$$

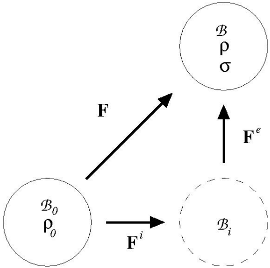
*Maxwell materials:*

$$\chi := \frac{1}{\tau} ||dev\boldsymbol{\sigma}||$$

$$\frac{\partial \overline{F}^e_{ij}}{\partial t} + \frac{\partial}{\partial x_k} \left( u_k \overline{F}^e_{ij} - u_i \overline{F}^e_{kj} \right) = -u_i \beta^e_j - \rho \Phi_{ij}$$

*where*

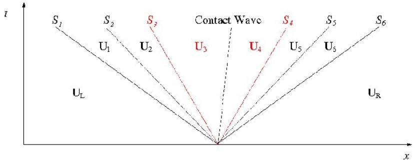
$$\boldsymbol{\Phi} = \mathbf{F}^e \mathbf{L}^i$$



# Solids: complete system

13+ equations:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_k} &= 0 \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \left(\rho u_i u_k - \sigma_{ik}\right)}{\partial x_k} &= 0 \\
\frac{\partial \overline{F}^e_{ij}}{\partial t} + \frac{\partial}{\partial x_k}\left(u_k \overline{F}^e_{ij} - u_i \overline{F}^e_{kj}\right) &= -u_i\beta_j - \rho\Phi_{ij} - \Psi(\rho, |\mathbf{F}^e|) \\
\frac{\partial \rho(\mathscr{E} + u^i u_i/2)}{\partial t} + \frac{\partial\left(\rho u_k(\mathscr{E} + u^i u_i/2) - u^i\sigma_{ik}\right)}{\partial x_k} &= 0
\end{aligned}
$$




Eigenstructure:

1. 7 wave families
2. 6 genuinely non-linear waves
3. 7 linear degenerate waves (speed of entropy wave)
4. Complete set of eigenvectors

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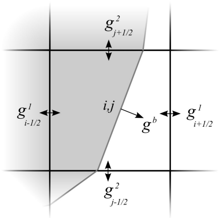
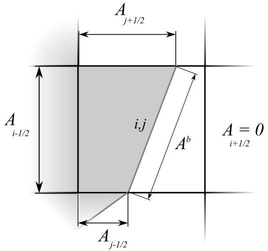
# Cut-cells: Finite volume discretisation

Each component assumed to have governing equations in form:

$$\mathbf{U}_t + \nabla \cdot \mathbf{F} = \mathbf{S}$$

Method of lines and finite volume discretisation:



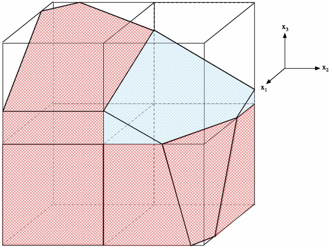

$$\begin{aligned} V_{ijk}^{\alpha,n+1}\mathbf{q}_{ijk}^{\alpha,n+1} &= V_{ijk}^{\alpha,n}\mathbf{q}_{ijk}^{\alpha,n} - \int_{t^n}^{t^{n+1}} \left[ \sum_{m=1}^{3} \left( A_{i_m+1/2}^{\alpha}\mathbf{g}_{i_m+1/2}^{\alpha,m} + A_{i_m-1/2}^{\alpha}\mathbf{g}_{i_m-1/2}^{\alpha,m} \right) \right. \\ & \qquad \left. + A_{ijk}^{\alpha,b}\mathbf{f}_{ijk}^{\alpha,b} - V_{ijk}^{\alpha,n}\mathbf{s}_{ijk}^{\alpha} \right] \mathrm{d}t \end{aligned}$$

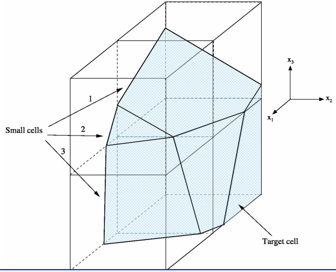
**Explicit Runge-Kutta used to solve time integral**

*For solids: use fractional stepping to address stiff inelastic source terms* $\mathbf{p}$

$$\begin{aligned} V_{ijk}^{\alpha,n+1}\mathbf{q}_{ijk}^{\alpha,\star} &= V_{ijk}^{\alpha,n}\mathbf{q}_{ijk}^{\alpha,n} - \int_{t^n}^{t^\star} \ldots \, \mathrm{d}t \\ \mathbf{q}_{ijk}^{\alpha,n+1} &= \mathbf{q}_{ijk}^{\alpha,\star} + \int_{t^\star}^{t^{n+1}} \mathbf{p}(\mathbf{q}_{ijk}^{\alpha,n+1}) \, \mathrm{d}t \end{aligned}$$

# Cut-cells: small cell problem




*For target 'T' and associated set $\mathcal{S}$ of small cells, final update:*

$$V_T^{(n+1)}\mathbf{q}_C^{(n+1)} = V_T^{(n+1)}\mathbf{q}_T^{(n+1)\star} - \frac{V_T^{(n+1)}\sum_{\mathcal{S}}(\mathbf{q}_S^{(n+1)\star}) - V_T^{(n+1)}\mathbf{q}_T^{(n+1)\star}\sum_{\mathcal{S}}V^{(n+1)}}{V_T^{(n+1)} + \sum_{\mathcal{S}}V^{(n+1)}}$$

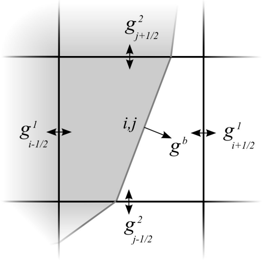
**Pairing method can have significant impact on symmetry preservation!**

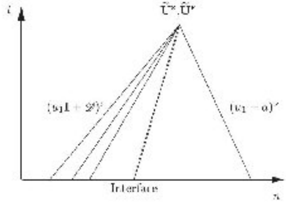
# Cut-cells: Coupling components

1. Rotate cell averaged values for each component onto normal to interface
2. Solve multi-material Riemann problem:

$$\mathbf{q} = \mathbf{q} + \mathbf{f}(\widetilde{\boldsymbol{\sigma}})$$

3. Rotate solution back
4. Compute interface fluxes using solution




Closure relations for various scenarios:

- Solid/solid
- Solid/vacuum
- Solid/fluid
- Solid/wall
- Fluid/fluid
- Fluid/wall

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# Interface tracking

Level-sets:

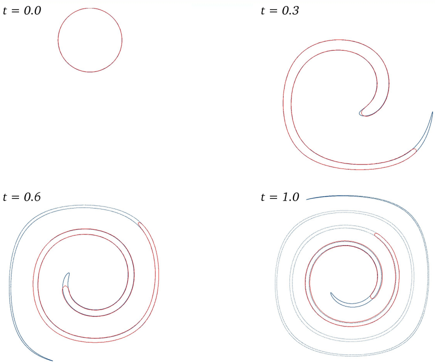
$$\phi_t + \mathbf{u}_I \cdot \nabla\phi = 0$$

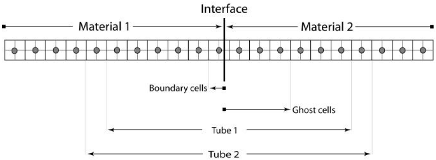
**Pros:**

- Allow slide
- Simplicity (geometry, advection)
- Allow breakup/merging
- Continuous representation of surfaces
- Allows use of RK method

**Cons:**

- Mass errors
- Cost? (extrapolation,reinitialisation)



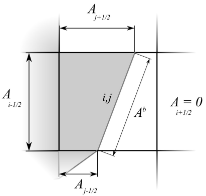



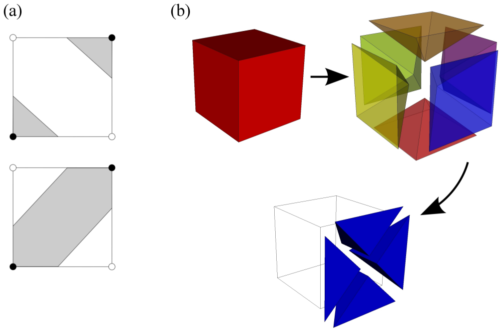
# Interface tracking

Polygonisation of zero-level-set:

(a) Marching cubes: fastest, but has ambiguous cases

(b) Marching tets: divide cube symmetrically into 24 tets; no ambiguity!



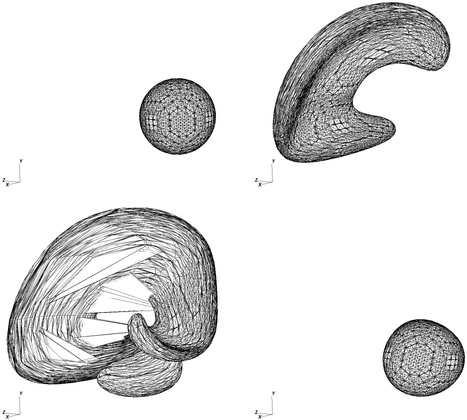

Provides:

- Material volume
- List of interface facets
- List of cell wall facets

Bitwise operations make this fast!

# Interface tracking

- Marching tets provides facet list representing interface
- Relatively straightforward (and cheap) to rebuild signed distance function from these
- Can then evolve the vertex list to represent interface
- As tri set becomes distorted, can rebuild as before (locally?)



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# High-order in space: WENO

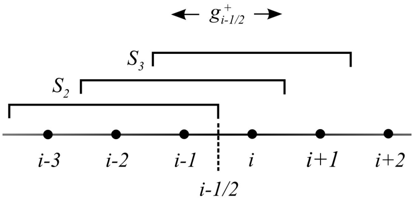
- Fluxes at cell faces; WENO-LLF:

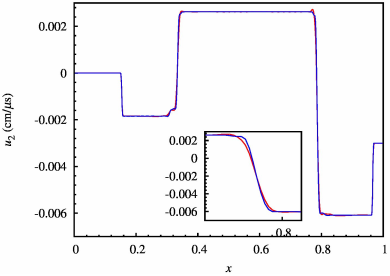
$$\mathbf{g}_{i-1/2} = \widetilde{\mathbf{g}}^{+}_{i-1/2} + \widetilde{\mathbf{g}}^{-}_{i-1/2}$$

where

$$\mathbf{g}^{+} = \frac{1}{2}\left(\mathbf{g}_L + \boldsymbol{\eta} \circ \mathbf{q}_L\right)$$

$$\mathbf{g}^{-} = \frac{1}{2}\left(\mathbf{g}_R - \boldsymbol{\eta} \circ \mathbf{q}_R\right)$$



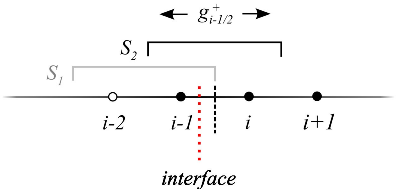

Choice of wave-speed:

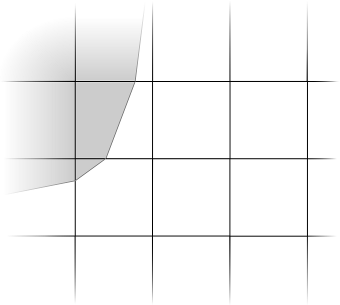
- basic single wave-speed
- local characteristic decomposition

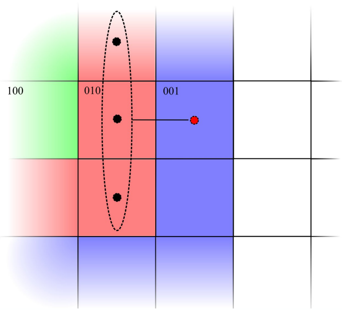
Characteristic analysis better; both expensive for solids!

# High-order in space: WENO

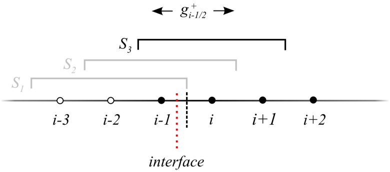
- Need to provide ghost states outside material regions to complete stencil
- Can extrapolate solution to multi-material Riemann problem
- In simplest case requires marching of interpolated values



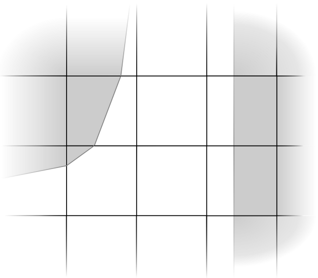



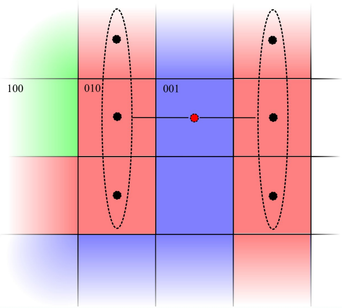


# High-order in space: WENO




- When boundaries of same material collide large stencils cause 'permeation' effect
- Effect exacerbated for larger stencils
- Adjust stencil to consider strips of material regions





# High-order in space: Hybrid WENO/centered

Use WENO:

- Around shocks and steep gradients of selected variables
- At the interfaces

**Extension of Hill *et al.*, JCP (2011)**

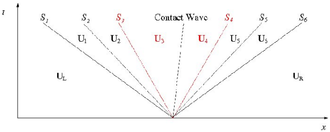
**Specifics:**

- 3rd Order TVD Runge-Kutta for time integration
- 5th Order WENO
- 6th Order central differences

Shock detection using Riemann based method of Lombardini (*M. Lombardini, PhD Thesis, Caltech, 2008*) adapted to solids:

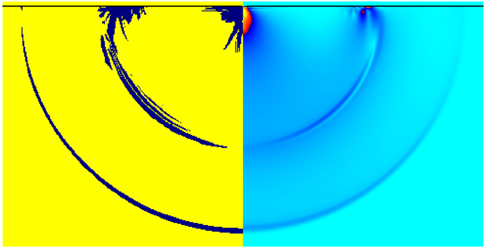
$$|u_R \pm \lambda_R^i| < |\widetilde{u} \pm \widetilde{\lambda}^i| < |u_L \pm \lambda_L^i|, \quad i = 1, 2, 3$$

Assuming $\widetilde{\cdot}$ to be Roe average works satisfactorily




Characteristic polynomial:

$$(u - \lambda)\det|\boldsymbol{\Omega} - (u - \lambda)^2| = 0$$



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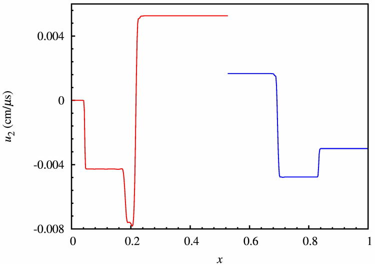
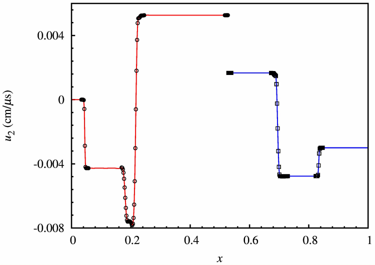
# 1d IVP

Solid/solid IVP from *Barton et al., JCP (2010)*

- Initial conditions:
  - Al on left (red)
  - Cu on right (blue)
  - initial interface at $x = 0.5$
- Slip boundary conditions
- Symbols indicate where WENO used

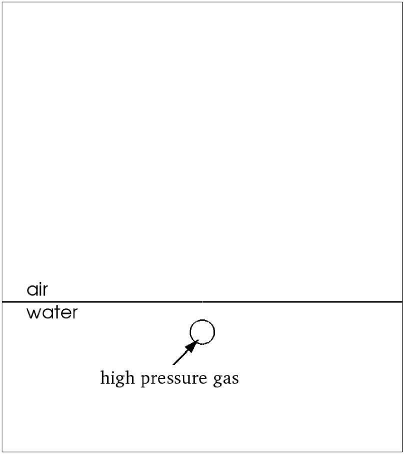


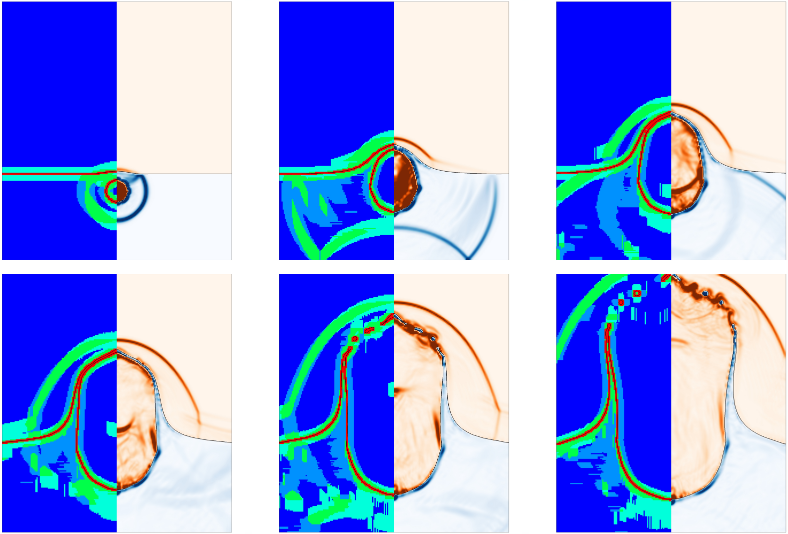
# 2d underwater explosion

To test the method in the event of large interface deformations consider the problem of an underwater explosion:

- Example from Liu *et al.*, JCP **215** (2006)
- Water modelled using stiffened gas EoS
- Air and high-pressure gas ideal $\gamma = 1.4$
- 1 level-set field

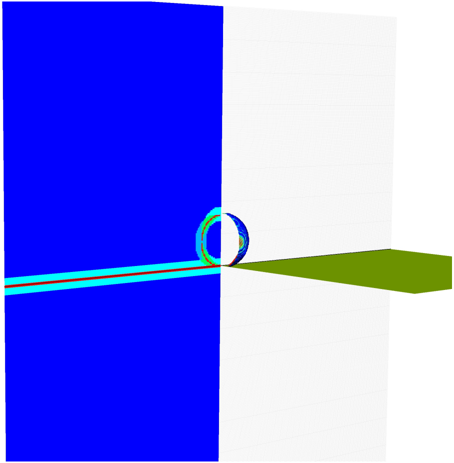


# 2d underwater explosion

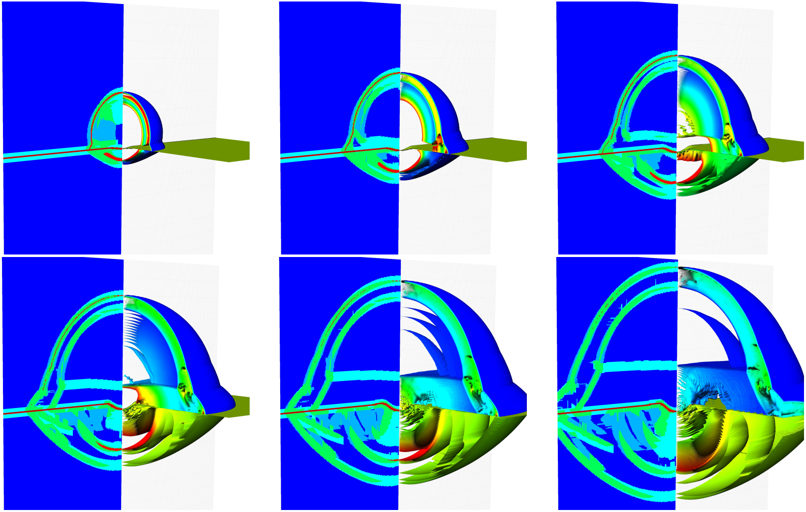


# 3d explosion in air

- Similar initial conditions to underwater explosion but with high pressure gas above the surface
- Solid deforms inelastically according to idealised plasticity with von-Mises yield surface
- Air and high-pressure gas ideal $\gamma = 1.4$



# 3d explosion in air



# Summary

Summary:

- A three-dimensional cut-cell method has been developed for coupled solid/fluid problems
- All variables are cell centred and grids remain fixed
- Method can handle largely distorting interfaces
- A hybrid method for the single component fluxes improves overheads, and paves the way for implementation of turbulence models for fluids

Future work:

- V&V; error analysis; cost analysis
- AMR to improve efficiencies
- incorporation of LES model (explicit LES)
- particle based front tracking to improve mass conservation
- improved constitutive models for solid materials